A MICROCONTINUUM APPROACH TO SOME BIO - FLUID DYNAMICAL PROBLEMS

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DOCTOR OF PHILOSOPHY

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D. PHILIP

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CERTIFICATE

It is certified that the work contained in the thesis entitled " **A MICROCONTINUUM APPROACH TO SOME BIO-FLUID DYNAMICAL PROBLEMS**" by Mr. D. Philip, for the award of Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree.

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DEDICATED

TO

MY GRAND FATHER
[Shri P.J.M. Pragasam Pillai (Late)]

for the uncompromising principles that guided his life

MY WIFE
(Smt. Sumathi Lourdhumary Philip)

for her magnificient devotion to her family,
who shouldered the responsibilities alone
and yet supported me all the way.

&

MY CHILDREN
(Evangeline Priya Philip & Philip Kevin)

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(D. PHILIP)

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SYNOPSIS

of the Ph.D. thesis on

A MICROCONTINUUM APPROACH TO SOME BIO-FLUID DYNAMICAL PROBLEMS

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In recent years study of human systems through interdisciplinary collaboration has led to the fascinating area of Bio-fluid dynamics. This subject seeks to understand the mechanism of living systems using the concepts of fluid dynamics. Since physiological system in itself is complex, several simplifications are made to analyse the problem. One such simplification is to treat bio-fluid as Newtonian fluid. In most of the work till to-date bio-fluid dynamical problems have been analyzed using classical Navier-Stokes equation. But physiological fluids in general behave like suspensions, e.g. blood is a suspension of red cells, white cells and platelets in plasma; cervical mucus is a suspension of macromolecules in a water like liquid etc. Various theories, e.g. microcontinuum fluid theory[Eringen (1964), Ariman et al. (1974)], continuum theory of mixtures [Atkin and Craine (1976)], etc., have been suggested to account for the suspension behaviour of fluids. This has motivated the work in this thesis where an attempt has been made to analyse some physiological flow problems using microcontinuum approach.

The following physiological flow situations have been analyzed in the thesis :

(i) flow through mild stenosis,

(ii) dispersion of a solute in a bio-fluid,

(iii) peristaltic transport, and

(iv) propulsion of spermatozoa in Cervical Canal.

We have studied the above mentioned problems by considering relevant bio-fluid to be of simple microfluid [Kang and Eringen (1976)] nature which accounts for micro deformation and micro rotation of the micro particles (suspended particles). In the last problem, we have also taken micropolar fluid model for mucus.

This thesis consists of 5 chapters.

Chapter I is of introductory nature and provides motivation for the present work done in this thesis. A survey of the pertinent literature is given so that the importance of the problems analysed in the subsequent chapters can be seen in right perspective. It also includes the basic equations governing the flow of a simple microfluid.

In Chapter II, we study the problem of flow through a stenosed artery. Here incompressible simple microfluid model for blood is assumed which account for the micro-deformation and micro-rotation of the particles. The flow is assumed to be steady

and laminar. The analysis has been made for very mild stenosis and the effect of cell free layer near the tube wall has been accounted in the model. The effects of various parameters on flow resistance (impedence) and wall shear stress have been discussed. It is noted that the increase in the resistance due to the stenosis height gets enhanced for certain combinations of simple microfluid parameters.

In Chapter III, the problem of dispersion of a chemically reactive solute in a simple microfluid is analysed considering the walls to be catalytic. The combined effects of homogeneous and heterogeneous chemical reaction is studied under Taylors limiting condition for flow through (i) channel and (ii) axi-symmetric tube. The effects of various parameters on equivalent dispersion coefficient have been discussed for both the cases.

Chapter IV deals with the peristaltic motion of simple microfluid through a distensible duct. The study consists of two parts (i) flow through a circular tube (ii) flow through a channel. In part I the analysis has been made for very long wavelength and hence the intertial terms in the equation of motion, governing the simple microfluid flow, have been neglected. In part II, low Reynolds number flow of a simple microfluid through a distensible channel has been considered, under the long wavelength approximation. The regular perturbation analysis is used and the expression for stream function upto

first order has been obtained. The variation of pressure drop and other flow characteristics with respect to the simple microfluid parameters have been discussed.

In Chapter V, we discuss the self-propulsion of spermatozoa through mucus filling the cervical canal, which is taken as a channel with flexible boundaries. The chapter consists of two parts : in part I, we have taken micropolar fluid model for mucus while in part II simple microfluid model has been considered. The spermatozoa is modelled as a two dimensional sheet which while swimming, sends down lateral waves of finite amplitude along its length. The model also considers the motion of the walls due to mascular activity which is represented by the transverse waves travelling along the flexible walls of the channel in the direction opposite the motion of the sheet. The analysis has been carried out for the inertia free flow under the assumption that the waves travelling along the channel walls and along the sheet are in synchronization under steady state and thus have same wavelength. The analysis presented here suggests that if the transverse waves are induced on the channel walls by some bio-chemical means, then it may be possible to reduce the speed of spermatozoa. This observation might be useful in fertility control.

In our study here, the effect of simple microfluid has been discussed by taking arbitrary and independent variations of the microcontinuum viscosity coefficients, since no experimental data

could be ascribed to the simple microfluid parameters. Hence, the results presented here must be used with caution. It has been observed that though the qualitative behaviour of the flow characteristics for a given simple microfluid is similar to those for Newtonian fluid, they show quantitative variations. The results show, reasonably well, the capabilities of microcontinuum approach in the physiological flow problems.

REFERENCES

ERINGEN, A.C., "SIMPLE MICROFLUIDS", Int. J. Engng. Sci., 2, 205-217, 1964.

ARIMAN, T., TURK, M.A., and SYLVESTER, N.D., "APPLICATION OF MICROCONTINUUM FLUID MECHANICS", Int. J. Engng. Sci., 12, 273-293, 1974.

ATKIN, R.J., and CRAINE, R.E., "CONTINUUM THEORIES OF MIXTURES : BASIC THEORY AND HISTORICAL DEVELOPMENT", Q. Jl. Mech. Appl. Math., 29, 209-244, 1976.

KANG, C.K., and ERINGEN, A.C., "THE EFFECT OF MICROSTRUCTURE ON THE RHEOLOGICAL PROPERTIES OF BLOOD", Bull. Math. Biology, 38, 135-159, 1976.

CHAPTER I

GENERAL INTRODUCTION

"If you believe that a vital field of science is a source of important mathematical problem, then the field of bio-science is the proper domain for you. The field of biology is so rich and complex that one can not visualize its being exhausted in the next one or two hundred years."

Richard Bellman

1.1 INTRODUCTION

In the last few decades attempts to understand the human system through interdisciplinary collaboration have led to the fascinating area of Bio-fluid dynamics. Bio-fluid dynamics seeks to understand the mechanism of living systems by using the concepts of fluid mechanics. Since the physiological systems, in general, are very complex, theoretical analyses have been made using several simplifying assumptions. In most of the work which has appeared to-date the relevent fluid is assumed to be Newtonian and problems have been treated using classical Navier-Stokes equations. However, it is observed that most of the bio-fluids behave as suspensions of deformable or rigid particles in a Newtonian fluid, e.g. blood is a macroscopically continuous suspension of particulate matters such as red cells, white cells and platelets in a microscopically continuous fluid (blood plasma) [(Houssay, 1955)]; Synovial fluid is a suspension of hyaluronic acid molecules in plasma [Chandra (1980)]; the

cervical mucus is a suspension of macromolecules in a water like liquid [Odeblad (1959, 1962)] etc.

In view of this some researchers have tried to account for the suspension behaviour of the bio-fluids by considering non-Newtonian models while still using the classical continuum approach. However, for a suspension, the particles contained in the fluid, may rotate under the velocity gradient of the flow field; thus, in addition to the linear momentum, a spin angular momentum is also introduced. This gives rise to the stress tensor becoming asymmetric, and hence the breakdown of the continuum hypothesis. Moreover, the classical approach becomes inadequate when the response of the body is sought to an external physical effect in which the length scale is comparable to the average molecular size. This limitation of the classical continuum approach has led to the development of theories of microcontinuum in which continuum media are now regarded as a set of structured particles which possess not only mass and velocity but also a substructure with which is associated a moment of inertia density and a micro-deformation tensor [Eringen (1964), Allen et al. (1967)]. These theories have been used to describe the mechanics of complex fluids like polymeric suspension, animal blood etc. and has found wide application in physiological and engineering problems [Cowin (1972), Ariman et al. (1974a), Kang and Eringen (1976)]. Therefore, in this thesis we have attempted to study the following physiological problems using the microcontinuum approach :

1. Flow through a stenosed artery,
2. Dispersion of a solute in a fluid,
3. Peristaltic transport in a duct,
4. Propulsion of spermatozoa in cervical canal.

In our study here, we have taken simple microfluid model [Eringen (1964)] for the bio-fluid in each of the problem. In addition to it the last problem has also been analysed by considering the motion of the micro-organism in micropolar fluid flow, through a channel.

We now give a brief introduction for simple microfluid and a brief survey of the literature pertaining to the above topics, so that the work done in this thesis can be seen in the proper perspective.

1.2 SIMPLE MICROFLUID

The fundamental concept of classical continuum mechanics is that all material bodies possess continuous mass densities, and the laws of motion and the axioms of constitution are valid for each part of the bodies regardless of its size. Thus, such an approach considers media as a dense collection of point masses which are void of internal structures. The theory becomes inadequate when more refined and complete description of material is sought. This necessitates the considerations of the intrinsic motions of material constituent and has led to the development of the theory of microcontinuum in recent years [Eringen and Suhubi (1964), Allen and Kline (1969), Kaloni and DeSilva (1969)]. A

review of various theories in this direction is given by Ariman et al. (1973a). The theory of microfluids, as introduced by Eringen (1964), deals with a class of fluids which exhibit certain microscopic effects arising from the local structure and micro motions of the fluid elements. These fluids can support stress moments and body moments and are influenced by the spin inertia. Thus, in the microstructured fluid model, material points of the body are considered to be small deformable bodies approximated by points with a triad of deformable directors attached to it.

In the theory of simple microfluids while many of the principles of classical continuum mechanics remain valid, they have to be augumented with additional balance laws and constitutive relations. For example, the presence of microscopic elements in a fluid gives rise not only to classical Cauchy stresses but also to couple stresses due to micro element interactions; also the orientation of the micro element as well as the internal angular momentum due to their rotational motion needs to be considered. These balance laws and constitutive relations have been derived by Eringen (1964). We reproduce, below, the balance equations and the constitutive relations governing the flow of a simple microfluid [Eringen (1964)]. (These equations are given here in rectangular coordinate system).

Conservation of mass :

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \qquad (1.1)$$

Balance of momentum :

$$t_{kl,k} + \rho\,(f_l - \dot{v}_l) = 0 \tag{1.2}$$

Balance of first stress moments :

$$t_{ml} - S_{ml} + \lambda_{klm,k} + \rho\,(l_{lm} - \dot{\sigma}_{lm}) = 0 \tag{1.3}$$

Conversion of micro-inertia :

$$\frac{\partial i_{km}}{\partial t} + i_{km,r}\,v_r - i_{rm}\,\nu_{rk} - i_{kr}\,\nu_{rm} = 0 \tag{1.4}$$

The constitutive equations for incompressible simple microfluid are

$$t_{kl} = -P\delta_{kl} + 2\mu d_{kl} + \zeta_1 \nu_{(kl)} + k\,(\nu_{[kl]} - \omega_{kl}) \tag{1.5}$$

$$s_{kl} = -P\delta_{kl} + 2\mu d_{kl} + 2\zeta_2 \nu_{(kl)} \tag{1.6}$$

$$\begin{aligned}\lambda_{klm} = {} & (\gamma_1\,\nu_{mr,r} + \gamma_2\,\nu_{rm,r})\,\delta_{kl} + (\gamma_4\,\nu_{lr,r} + \gamma_5\,\nu_{rl,r})\,\delta_{km} \\ & + (\gamma_7\,\nu_{kr,r} + \gamma_8\,\nu_{rk,r})\,\delta_{lm} + \gamma_{10}\,\nu_{kl,m} + \gamma_{11}\,\nu_{km,l} \\ & + \gamma_{12}\,\nu_{lk,m} + \gamma_{13}\,\nu_{mk,l} + \gamma_{14}\,\nu_{lm,k} + \gamma_{15}\,\nu_{ml,k}\end{aligned} \tag{1.7}$$

where

ρ = mass density,	i_{km} = microinertia tensor,
v_k = velocity vector,	t_{kl} = stress tensor,
f_l = body force per unit mass,	$\dot{\sigma}_{lm}$ = spin inertia,

s_{kl} = micro stress average tensor, ν_{kl} = gyration tensor,

λ_{klm} = the first stress moment tensor,

l_{lm} = the first body moment per unit mass,

$$\omega_{kl} = \frac{1}{2}(v_{k,l} - v_{l,k}) \; ; \; d_{kl} = \frac{1}{2}(v_{k,l} + v_{l,k}) \qquad (1.8)$$

and γ_1 to γ_{15}, μ, ζ_1, ζ_2 and k are viscosity coefficients. Here partial differentiation is denoted by (,) and material time derivative by a superposed (.).

The symmetric part of the gyration tensor, $\nu_{(kl)}$ represents micro-deformation while the skew symmetric part, $\nu_{[kl]}$ gives micro-rotation of the microelements.

In our study in the subsequent chapters, we restrict our attention to two-dimensional motion of incompressible simple microfluid where the body forces and body couples are neglected. Thus, the velocity $\bar{V}$, gyration tensor ,ν'_{kl}, pressure P, have the forms

$$\vec{V} = (U\,(X,Z)\,,\, 0\,,\, W\,(X,Z))$$

$$\nu'_{kl}(X,Z) = \begin{cases} \nu'_{(13)} + \nu'_{[13]} & \text{when } k=1,\ l=3 \\ \nu'_{(13)} - \nu'_{[13]} & \text{when } k=3,\ l=1 \\ 0 & \text{otherwise.} \end{cases}$$

$$P = P(X,Z)$$

Thus the balance equations for incompressible fluid flow reduce to

$$-\frac{\partial P}{\partial X} + \mu\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right]U + \frac{\zeta_1 \partial \nu'_{(31)}}{\partial Z} + k\left[\frac{\partial \nu'_{[31]}}{\partial Z} - \frac{1}{2}\frac{\partial^2 W}{\partial X \partial Z} + \frac{1}{2}\frac{\partial^2 U}{\partial Z^2}\right]$$

$$= \rho\left[\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z}\right] \tag{1.9}$$

$$-\frac{\partial P}{\partial Z} + \mu\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right]W + \frac{\zeta_1 \partial \nu'_{(13)}}{\partial X} + k\left[\frac{\partial \nu'_{[13]}}{\partial X} - \frac{1}{2}\frac{\partial^2 U}{\partial X \partial Z} + \frac{1}{2}\frac{\partial^2 W}{\partial X^2}\right]$$

$$= \rho\left[\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial X} + W\frac{\partial W}{\partial Z}\right] \tag{1.10}$$

$$(K_1-K_3)\frac{\partial^2(\nu'_{31})}{\partial X^2} + (K_2-K_3)\frac{\partial^2(\nu'_{13})}{\partial X^2} + (K_1+K_3)\frac{\partial^2(\nu'_{13})}{\partial Z^2} + (K_2+K_4)\frac{\partial^2(\nu'_{31})}{\partial Z^2}$$

$$+ 2(\zeta_1-2\zeta_2)\nu'_{(13)} + 2k\left[\nu'_{[13]} - \frac{1}{2}\frac{\partial U}{\partial Z} + \frac{1}{2}\frac{\partial W}{\partial X}\right] = 0 \tag{1.11}$$

$$(K_1+K_3)\frac{\partial^2(\nu'_{31})}{\partial X^2} + (K_2+K_3)\frac{\partial^2(\nu'_{13})}{\partial X^2} + (K_1-K_3)\frac{\partial^2(\nu'_{13})}{\partial Z^2} + (K_2-K_4)\frac{\partial^2(\nu'_{31})}{\partial Z^2}$$

$$+ 2(\zeta_1-2\zeta_2)\nu'_{(31)} + 2k\left[\nu'_{[31]} - \frac{1}{2}\frac{\partial W}{\partial X} + \frac{1}{2}\frac{\partial U}{\partial Z}\right] = 0 \tag{1.12}$$

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0 \tag{1.13}$$

where

$$\left.\begin{aligned}
K_1 &= \gamma_1 + \gamma_{13} + \gamma_{15} + \gamma_4 + \gamma_{12} + \gamma_{14} \geq 0 \\
K_2 &= \gamma_2 + \gamma_{11} + \gamma_{14} + \gamma_5 + \gamma_{10} + \gamma_{15} \geq 0 \\
K_3 &= \gamma_1 + \gamma_{13} + \gamma_{15} - (\gamma_4 + \gamma_{12} + \gamma_{14}) \leq 0 \\
K_4 &= \gamma_2 + \gamma_{11} + \gamma_{14} - (\gamma_5 + \gamma_{10} + \gamma_{15}) \geq 0
\end{aligned}\right\} \qquad (1.14)$$

To make an order of magnitude analysis, we nondimensionalize the various quantities as follows :

$$(u,w) = (U,W)/V^* \ , \ \nu_{ij} = \frac{h}{V^*}\nu'_{ij} \ , \ p = Ph/(\rho V^{*2}L)$$

$$x = X/L \ , \ z = Z/h, \quad \bar{t} = V^* t/L, \quad R^* = \rho h V^*/\mu,$$

$$(E_1,E_2,E_3) = (-\zeta_1, \zeta_2, k)/\mu, \quad \bar{K}_i = \frac{K_i}{\mu}h^2 \quad \varepsilon^* = h/L$$

(where V^* is characteristic velocity, L^* is characteristic length along the axis of the channel, h is height of the channel).

Thus, we get :

$$-\frac{\partial p}{\partial x} + \left[\varepsilon^{*2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]u - E_1\frac{\partial\nu_{(13)}}{\partial z} + E_3\left[-\frac{\partial\nu_{[13]}}{\partial z} - \frac{\varepsilon^*}{2}\frac{\partial^2 w}{\partial x\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}\right]$$

$$= R^*\varepsilon^*\left[u\frac{\partial u}{\partial x} + \varepsilon^* w\frac{\partial u}{\partial z}\right] \qquad (1.15)$$

$$-\varepsilon^{*}\frac{\partial p}{\partial z}+\left[\varepsilon^{*2}\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial z^{2}}\right]w-E_{1}\varepsilon^{*}\frac{\partial \nu_{(13)}}{\partial x}+E_{3}\varepsilon^{*}\left[\frac{\partial \nu_{(13)}}{\partial x}-\frac{1}{2}\frac{\partial^{2}u}{\partial x\partial z}+\frac{1}{2}\varepsilon^{*}\frac{\partial^{2}w}{\partial x^{2}}\right]$$

$$= R^{*}\varepsilon^{*}\left[u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right] \qquad (1.16)$$

$$(\bar{K}_{1}-\bar{K}_{3})\varepsilon^{*2}\frac{\partial^{2}\nu_{31}}{\partial x^{2}}+(\bar{K}_{2}-\bar{K}_{4})\varepsilon^{*2}\frac{\partial^{2}\nu_{13}}{\partial x^{2}}+(\bar{K}_{1}+\bar{K}_{3})\frac{\partial^{2}\nu_{13}}{\partial z^{2}}+(\bar{K}_{2}+\bar{K}_{4})\frac{\partial^{2}\nu_{31}}{\partial z^{2}}$$

$$-2(E_{1}+2E_{2})\,\nu_{(13)}+E_{3}\left[2\nu_{[13]}-\frac{\partial u}{\partial z}+\varepsilon^{*}\frac{\partial w}{\partial x}\right]=0 \qquad (1.17)$$

$$(\bar{K}_{1}+\bar{K}_{3})\varepsilon^{*2}\frac{\partial^{2}\nu_{31}}{\partial x^{2}}+(\bar{K}_{2}+\bar{K}_{4})\varepsilon^{*2}\frac{\partial^{2}\nu_{13}}{\partial x^{2}}+(\bar{K}_{1}-\bar{K}_{3})\frac{\partial^{2}\nu_{13}}{\partial z^{2}}+(\bar{K}_{2}-\bar{K}_{4})\frac{\partial^{2}\nu_{31}}{\partial z^{2}}$$

$$-2(E_{1}+2E_{2})\,\nu_{(13)}+E_{3}\left[-2\nu_{[13]}-\varepsilon^{*}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]=0 \qquad (1.18)$$

$$\frac{\partial u}{\partial x}+\varepsilon^{*}\frac{\partial w}{\partial z}=0 \qquad (1.19)$$

Here it can be noted that for the inertia free flow ($R^{*}=0$) right hand side terms in eqns. (1.14) and (1.15) can be dropped. Further, for the case when h << L (i.e. infinite characteristic length) terms containing ε^{*} can be dropped.

1.2.1 Micropolar Fluid

A subclass of simple microfluid which exhibits only the micro-inertial effects and micro-rotational inertia is known as micropolar fluid. In micropolar fluids the directors attached to

the points are taken to be rigid which implies that the microstructure have only micro-rotations and no microstretchings and shearings. For the sake of completeness we give below the constitutive equations for an incompressible micropolar fluid as given by Eringen (1966).

$$T_{ij} = -p\delta_{ij} + (\mu + \frac{\chi}{2}) (v_{i,j} + v_{j,i}) + \chi v_{j,i} - \chi \epsilon_{ijk} \nu_k \qquad (1.20)$$

$$M_{ij} = \beta \nu_{i,j} + \gamma \nu_{j,i} \qquad (1.21)$$

where T_{ij} and M_{ij} are stress tensor and couple stress tensor respectively, δ_{ij} is the Kronecker delta, ϵ_{ijk} an alternating tensor, p the isotropic pressure and ν_k the micro-rotation vector, μ is the viscosity coefficient of the classical mechanics and χ is the new viscosity coefficient for the micropolar fluid. Comma after a suffix denotes partial differentiation.

The balance equations governing the flow of incompressible micropolar fluid are [Eringen, 1966]:

Conservation of mass :

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 . \qquad (1.22)$$

Conservation of linear momentum :

$$(\mu+\chi/2) v_{k,ll} + \chi \epsilon_{klm} \nu_{m,l} - p_{,k} + \rho(f_k - \dot{v}_k) = 0 \qquad (1.23)$$

Conservation of angular momentum :

$$\gamma \nu_{k,ll} + \chi \varepsilon_{klm} v_{m,l} - 2\chi \nu_k + \rho(l_k - j \dot{\nu}_k) = 0 \qquad (1.24)$$

where f_k is the body force per unit mass, l_k is the body couple and j is the micro inertia constant. Superposed dot indicates the material derivative.

1.2.2 Boundary Conditions

The determination of proper boundary conditions for microstretch and microrotation is a complicated topic, and a well posed problem requires a proof of uniqueness theorem which is not available even in the case of fluid dynamics [Kang and Eringen (1976)]. While the no slip conditions are assumed for the velocity components at the boundary surfaces, various types of conditions have been suggested for the microrotation [Migun (1984), Kirwan (1986)]. Thus, for the skew symmetric part of the gyration tensor one can assume :

(a) perfect adherence at the rigid wall i.e. $\nu_{[kl]} = 0$

(b) moment free condition [Eringen (1966)]

i.e. $\lambda_{[kl]m}\, n_m = 0$ at the wall

(c) given fixed gyration at the wall [Ariman (1971)]

i.e. $\nu_{[kl]} = \nu_o$, a constant

(d) $\nu_{[kl]} = s w_{kl}$ where $0 \le s \le 1$ [Aero et al. (1965)]

However, not much has been said about the micro-deformation of particles at the boundary surfaces. It seems that the

selection of suitable boundary conditions for the simple microfluid depends upon the type of fluid and the physical situation. In our work, we have assumed the following type of conditions as suggested by [Kang & Eringen (1976)] for a suspension.

$$\nu_{(13)} = A_2 d_{13} \quad \text{and} \quad \nu_{[13]} = A_3 w_{13} \tag{1.25}$$

A particular case of $A_2 = A_3 = 0$ refers to a perfect adherence condition and denote that the fluid solid interaction is so strong that it does not allow the rotation and deformation of the microstructure at the wall. It represents the extreme case in which the walls act to completely inhibit microrotation and micro-deformation.

1.3 STENOSIS

The word 'STENOSIS' means narrowing of the lumen of a tube or orifice. Thus, an arterial stenosis refers to a narrowed or constricted segment of an artery. An arterial stenosis is also referred to as 'atherosclerosis'. This can cause serious circulatory disorders by reducing or occluding the blood supply to an organ; e.g., stenosis in coronary artery can cause myocardial infarction leading to heart failure, likewise in carotid artery it can bring about cerebral strokes.

Though the process of initiation of arterial stenosis is not completely known, it is well recognised that many factors are involved in the development of stenosis, e.g. hypertension,

intake of more lipid rich diet, lack of oxygen and lack of physical exercise [Spain (1966)]. It has been suggested that hydrodynamic factors such as wall shear stress, pressure distribution, separation phenomena etc. play an important role in the initiation and early progression of the stenosis [Rodkiewicz (1974), Stehbens (1982), Narem & Levesque (1987)] . Thus, in the recent years researchers have shown enormous interest to study blood flow through stenotic region. One of the earliest experimental study reported in the literature is attributed to Mann et al. [1938], who studied blood flow in a decreasing lumen of an artery. Young [1968] was the first one to study the problem mathematically. He considered a simplified linear model of flow in a mildly constricted tube, with the following geometry of the stenosis

$$R^*(Z) = \frac{R(Z)}{R_o} = \begin{cases} 1 - \dfrac{\delta_s}{2R_o}\left\{1 + \cos\dfrac{2\pi}{L_o}\left(Z - d - \dfrac{L_o}{2}\right)\right\} \\ 1 \quad \text{otherwise} \end{cases} \tag{1.26}$$

where $R(Z)$ is the radius of the artery in the stenotic region, R_o is the constant radius elsewhere, L_o is the length of the stenosis and δ_s is its maximum height. Stenosis location is indicated by ,d, and Z the axial coordinate. Since then much attention has been given to the investigations of the steady/unsteady flow through arterial stenosis both analytically as well as experimentally [Forrester & Young (1970), Lee and Fung (1970), Caro et al. (1971), Cheng et al. (1972, 1973), Morgan and

Young (1974), Roth et al. (1976), Daly (1976), Deshpande (1976), Young (1979), McDonald (1979), Shukla et al. (1980a,b), Williams and Javadpour (1980) and Chaturani and Swamy (1983)]. Shukla et al. (1980c) have considered power-law and casson models for blood and have concluded that this non-Newtonian behaviour of the blood is helpful in the functioning of diseased arterial circulation. In recent years various micro-continuum models have been suggested to consider the suspension behaviour of blood. Hence, Sinha & Singh (1984), Srivastava (1985) have analyzed the effect of couple stress on the blood flow through a thin artery. They have shown that an increase in the couple stress parameter increases the resistance to the flow and the wall shear stress. Parvathamma and Devanathan (1983) have considered the steady flow of micropolar fluid through stenosed tube. They have shown that for a given Reynolds number, the shearing stress reaches a maximum immediately preceding the throat and then rapidly decreases in the diverging section. They have also shown that as the Reynolds number increases, the point of maximum shear stress moves upstream due to the slope of the tube wall. Recently Pralhad and Schultz (1988) have studied the stenosis problem for different diseases, considering blood as a polar fluid. Recently Sagayamary and Devanathan (1989) have studied the flow of couple stress fluid through tubes of varying cross-sections and have applied the results to the blood flow.

Further it has been noted [Hershey et al. (1966), Middleman (1972), Lih (1975)], that when blood flows through arteries, near

the wall of the arteries, a cell free layer is formed i.e. there exist a plasma layer near the wall and the cells are radially distributed. Shukla et al. (1980a,b) have studied the effect of peripheral layer on the flow of blood to take into account the clustering of cells in the central zone and have concluded that such a layer helps the flow in a diseased system.

Keeping in mind the above mentioned points, we consider in Chapter II, the blood flow through stenotic artery using micro-continuum approach. Blood is considered to be simple microfluid as suggested by Kang and Eringen [1976]. The effect of cell free layer near the wall is also considered.

1.4 DISPERSION IN BIOLOGICAL FLUID FLOWS

The process of dispersion is important in chemical as well as in biological systems. In particular, in all living beings matters like nutrients and metabolic products are transporated by blood as a result of diffusive and convective mechanisms; drug is transported as a result of diffusion through blood. Also the process of dispersion of a solute matter in a flow system is an important phenomenon in rivers, in oil pipelines, in hydrualic industrial devices and various chemical systems. Thus, various investigators have studied dispersion/diffusion under steady and unsteady conditions in different physical systems.

Taylor (1953) was the first one to mathematically formulate the problem of dispersion. He used his model to measure the rate

of flow of blood in the arteries of animals by injecting a small volume of highly conducting liquid at some point. In his classical paper, Taylor (1953) discussed the dispersion of a soluble matter in a viscous fluid flowing in a ciricular pipe under laminar conditions. He found that relative to a plane moving with the mean speed of the flow, the solute is dispersed with an apparent diffusion coefficient $R^2\bar{v}_x^2/48D$ (where R, $\bar{v}_x$ and D are the radius of the pipe, the average velocity and the molecular diffusion coefficient respectively). In a subsequent study, Taylor (1954) showed that dispersion of solute can be used to measure molecular diffusion provided :

$$\frac{4L}{R} \gg \frac{\bar{v}_x R}{D} \gg 6.9$$

(L being a given length along the flow direction). Aris (1956) while extending the Taylor's work, found that the rate of growth of variance of the solute distribution is proportional to the sum of the molecular diffusion and Taylor diffusion coefficient.

In the early studies it was assumed that the solute is chemically non-reactive. However, in a variety of problems in chemical engineering, diffusion of solute takes place in the presence of an irreversible first order chemical reaction, e.g. hydrolysis of ester, gas absorption in an agitated tank with chemical reaction [Bird et al. (1960)]. It is also suggested that the walls of the tube may be catalytic and hence may give rise to heterogeneous chemical reaction at the surface e.g., tubular flow reactors with heterogeneous catalysts. In view of this, Cleland

and Wilhelm (1956) have investigated the problem of a finite first order homogeneous reaction in a pipe under laminar flow conditions by a finite difference method and supproted their theoretical results with experimental data. Katz [1959] considered the chemical reaction catalyzed on the wall of the tube and discussed the heterogeneous reaction taking place at the wall on the concentration profiles. Later many generalizations/ extensions of these studies have been attempted to study dispersion of a solute in Newtonian and non-Newtonian fluid flows [Cumming et al. (1966), Gill et al. (1967, 1972), Ghosal (1971), Scherer et al. (1972), Gupta & Gupta (1972), Dutta et al. (1974), Shukla et al. (1979), Shukla & Parihar (1980d), Chandra & Agarwal (1983), Das & Mazumder (1989) etc.].

Gill et al. in a series of papers (1967, 1970, 1972) have developed an exact method of analysis of unsteady convective diffusion system. Taylor's approach has been used to study the dispersion of a solute in non-Newtonian fluids by Fan and Hwang (1965), Fan and Wang (1966), Ghosal (1971), Shah and Cox (1974) for non reactive solute, while Dutta et al. (1974), and Shukla et al. (1979) considered dispersion of a solute in the presence of a homogeneous chemical reaction. Shukla et al. (1980d, 1982a) have studied the effect of peripheral layer on the Taylor dispersion coefficient and have found that the Taylor coefficient decreases as the viscosity or peripheral layer fluid decreases. Combined first order heterogeneous and homogeneous reactions were studied by Walker (1961), Solomon & Hudson (1967), Gupta & Gupta (1972)

and others. Gupta and Gupta (1972) in particular have considered unsteady dispersion of a solute with simultaneous chemical reaction in a liquid flowing in a channel under laminar conditions using Taylor's approach. Barton (1984) has used the Aris method of moments to study the dispersion of reactive contaminant in pipe Poisuille flow in the presence of first order reactions at the wall.

Using microcontinuum approach, Soundalgekar (1970) has analysed dispersion of a solute in the laminar fully developed flow of micropolar fluids flowing in a channel. In a subsequent paper Soundalgekar and Chaturnai (1980), studied the effect of couple stresses on the dispersion of a soluble matter in a pipe flow. Chandra and Agarwal (1983) studied the dispersion of a solute matter in simple microfluids flowing through channel and pipe under Taylor's limiting condition. They have considered the first order chemical reaction in fluid but did not consider the walls to be catalytic. Hence, in Chapter III, the dispersion of a solute in a simple microfluid is analysed considering the walls to be catalytic and the combined effect of homogeneous and heterogeneous reactions is considered.

1.5 PERISTALSIS

The word peristalsis is derived from the Greek word "Peristaltikos", which means clasping and compressing. It may be described as the process of fluid being transported through a distensible tube as contraction and expansion waves propagate

along its length on the boundary walls. Its main function is to propel fluid contents in the tube. It plays an indispensible role in transporting many physiological fluids inside living bodies, e.g. urine transport from kidney to bladder through ureter, in bile duct, movement of ovum in fallopian tubes, transport of spermatozoa in ductus efferentes of the male reproductive tract and in cervical canal of the female reproductive system, vasomotion of small blood vessels, mixing the contents inside the gastro-intestinal tract etc. Many bio-mechanical and engineering devices have also been designed on the basis of the principle of peristaltic pumping.

From fluid mechanical point of view, peristaltic pumping is characterized by the dynamic interaction of fluid flow with movement of flexible boundaries. Hence several authors have studied peristaltic transport of fluids from fluid dynamical as well as physiological points of view. Excellent reviews of the literature in this direction have been given by Jaffrin and Shapiro (1971), Rath (1980), Srivastava & Srivastava (1984).

In particular Burns and Parkes (1967) have discussed the peristaltic motion in both two dimensional and axisymmetric cases under the assumption of small Reynolds number and linearized boundary conditions with perturbation technique. Barton and Raynor (1968) studied peristaltic motion in a circular tube with reference to small intestine using long as well as short wavelength approximations. Fung and Yih (1968) considered an idealized two dimensional analogy of the real problem and studied

the problem of peristaltic waves induced by a sinusoidal travelling wave motion of the walls at moderate amplitudes. Later this was extended by Yin and Fung (1969) to axisymmetric flow in a circular tube by perturbation method of solutions in terms of the ratio of the wave amplitude to tube radius. They had also taken into consideration the nonlinear convective terms. Chow (1970) obtained the solution of the peristaltic transport in a circular cylindrical tube in the form of power series expansion when the Reynolds number is small. Mittra and Prasad (1974) studied the interaction of Poiseuille flow with peristaltic motion. Gupta and Seshadri (1976) have analysed the peristaltic transport in a non-uniform tube. Lykoudis and Roos (1970) studied the fluid mechanics of the ureter, from a lubrication theory point of view. Liron (1976) has presented a complete solution for peristaltic flow in a pipe and in a channel by means of an expansion in the Reynolds number and wave number for arbitrary wave shape. Gopalan and Devanathan (1983) have studied the pressure variation in a peristaltic motion with arbitrary wave shape when a catheter is inserted.

In all these studies the effect of viscosity variation of the fluid has not been taken into account, which may play an important role in some physiological situations. To highlight this effect of peripheral layer, Shukla et al. (1980e) studied peristaltic transport of a Newtonian fluid with viscosity varying across the tube, under long wavelength approximation. The analysis has been applied and compared with the observed flow

rates in intestine and ductus efferentes of male reproductive tract, and the importance of the peripheral layer viscosity has been pointed out.

Though it has been found experimentally that biofluids (e.g. human feces, bile, mucus etc) behave rheologically as non-Newtonian fluid [Odebald (1959, 1962), Han and Barnett (1973), Patel et al. (1973), Rodkiewicz et al. (1979)], only a few researchers have paid attention to study the peristaltic transport of non-Newtonian fluid. Raju and Devanathan (1972) studied the peristaltic flow of a power-low fluid. They have obtained power series solution for the stream function and the velocity components in terms of the amplitudes of deformation. They have also discussed the influence of the applied pressure gradient along with the non-Newtonian parameter on the stream lines. Picologlou et al. (1973) studied the peristaltic motion of human faeces characterized by a power-law model from the point of view of pressure generation in the contracting part and concluded that the peristalsis, in colon, though rare, is a major propulsive mechanism. Mittra and Prasad (1974) extended the work of Fung and Yih (1968) to study the effect of elastic or visco-elastic wall on peristaltic flow. They have found the existence of "mean flow reversal", both at the centre and at the boundaries. Raju and Devanathan (1974) extended their previous work to include the discussions about the peristaltic motion of the visco-elastic fluid. Radhakrishnamacharya (1982) has analysed peristaltic transport of a power-law fluid. In view of the

power-law behaviour of the chyme and the presence of peripheral mucus layer in intestines, Shukla et al. (1982b) studied the peristaltic transport of a power-law fluid whose consistency varies along the radial direction and investigated the effects of the peripheral layer consistency and the pseudoplastic nature of the bio-fluid on the flow characteristics. Kaimal (1978), Srivastava & Srivastava (1989) have studied the peristaltic motion of a suspension of rigid particles in a tube of arbitrary wave shape, at low Reynolds number under long wavelength approximation. An asymptotic method was developed for the solution of peristaltic transport of a viscous fluid with solid particles in a flexible tube of arbitrary cross-section by Shen et al. (1981). Böhme and Friedrich (1983) considered the mechanism of peristaltic transport of an incompressible visco-elastic fluid by means of an infinite train of sinusoidal waves travelling along the wall of the duct, in the case of a plane flow. Peristaltic flow of power-law non-Newtonian fluid containing spherical particles which are neutrally buoyant, is studied by Rath (1983). Devi and Devanathan (1975) have analyzed peristaltic motion of micropolar fluid under long wavelength approximation. Srivastava (1986) studied the peristaltic transport of a couple stress fluid under zero Reynolds number and long wavelength approximation. He compared the results with those for Newtonian fluid model to show that the magnitude of pressure rise under a given set of conditions is greater in the case of couple stress fluid. Therefore, in Chapter IV we consider the flow of simple microfluid in a circular tube and in a channel

so as to account for the microstretchings, shearings and micro-rotations.

1.6 SWIMMING OF MICRO-ORGANISMS

The study of self-propelling micro-organisms in liquids has received considerable attention in the recent years due to its physiological applications such as motion of spermatozoa in cervical mucus. Spermatozoa are the tiny living cells (having length $5\text{x}10^{-3}$ cm, diameter 10^{-5} cm) which have their own motility. They move towards the oviduct through the mucus, filling the cervical canal, by sending waves of lateral displacement down the tails. Its propulsion resembles to that of fish and its direction of movement is opposite to that of the propagation of the waves.

Since the pioneering work of Taylor (1951) who studied the propulsion of spermatozoa by modelling it as a two dimensional infinite extensible sheet with sinusoidal wave travelling down its length, several researchers have analyzed the motion of self propelling micro-organisms [Hancock (1953), Reynolds (1965), Lighthill (1974), Shack and Lardner (1974), Shen et al. (1975), Pirronneau (1975), Rubinow (1975), Shukla et al. (1978), Sinha et al. (1982), Parihar (1983), Blake et al. (1983) Shukla et al. (1988)].

In particular Gray and Hancock (1955) obtained the velocity of propulsion of sea urchin spermatozoa with remarkable accuracy using simple dynamical arguments. The earlier studies considered

movement of isolated sheets in unbounded domain. Lately some studies have been made to analyze the hydrodynamics of a swimming spermatozoa in a bounded fluid which may actually be the case once the sperm enters the cervical canal. Reynolds (1965) considered two dimensional model and used Taylor's sheet approach to consider the effect of a rigid wall adjacent to the sheet. This analysis is based on the assumption that the amplitude of the tail motion is small. He observed that the transport of sperm through the cervical mucus is too rapid to be accounted for by the invitro swimming speed of approximately 50 μm/sec of sperm in cervical mucus. Lighthill (1975) presented a general theory of swimming of micro-organisms at low Reynolds numbers. Shukla et al. (1978) studied the swimming of spermatozoa in cervix by considering the interaction of the micelles aligned along the wall. They have also pointed out that the propelling velocity of the sperm increases due to the dynamical interaction of the cervix and due to the presence of the peripheral layer. Attempts by other researchers were also made to explain the motion of spermatozoa in the female genital tract by considering the dynamical interaction of the wall. Smelser et al. (1974) investigated the swimming of sperm in an active channel in which the walls of the channel also vibrate. Blake et al. (1983) presented a theoretical model of ovum transport in the oviduct incorporating transport mechanisms due to ciliary and mascular activity of the wall by adding a force distribution term in the equation of motion. The contraction and relaxation of smooth muscles of uterus and the oviduct may give rise to peristaltic

motion of the channel wall. This effect was considered by Shukla et al. (1988).

There is a strong likelihood that the motion of spermatozoa through the cervical canal may be studied from a micro-continuum view point. It has been observed that cervical mucus is a suspension of high molecular weight macromolecules in a water like liquid with viscosity 0.03P; Odebald (1959, 1962). In luteal phase the mucus resembles like a close mesh, having a spacing of 0.03 μm in them, Elstein (1971), Davajan (1971). This motivated Sinha et al. (1982) to study the propulsion of spermatozoa in a micropolar fluid. They have considered the channel walls to be rigid. Thus in Chapter V, swimming of micro-organism is studied. We mathematically analyze the motion of the spermatozoa in the cervical canal by considering the transverse waves along its tail and peristaltic motions along the cervical wall. Cervical mucus is considered as micropolar fluid in Part I and as a simple microfluid in Part II.

1.7 PRESENT STUDY

In view of the above, we present here a microcontinuum approach to study some physiological flow problems. This approach accounts for the microrotations and microdeformations of the suspended particles.

In chapter II, we study the problem of flow through a stenosed artery. Here incompressible simple microfluid model for blood is assumed. The flow is assumed to be steady and laminar.

The analysis has been made for very mild stenosis and the effect of cell free layer near the tube wall has been accounted in the model. The effects of various parameters on flow resistance (impedance) and wall shear stress have been discussed. It is noted that the increase in the resistance due to the stenosis height gets enhanced for certain combinations of simple microfluid parameters.

In chapter III, the problem of dispersion of a chemically reactive solute in a simple microfluid is analysed considering the walls to be catalytic. The combined effects of homogeneous and heterogeneous chemical reaction is studied under Taylor's limiting condition for flow through (i) channel and (ii) axi-symmetric tube.

Chapter IV deals with the peristaltic motion of simple microfluid through a distensible duct. The study consists of two parts (i) flow through a circular tube (ii) flow through a channel. In part I, the analysis has been made for very long wavelength by neglecting inertial terms. In part II, low Reynolds number flow of a simple microfluid through distensible channel has been considered, under the long wavelength approximation. The regular perturbation analysis is used and the expression for stream function upto first order has been obtained.

In Chapter V, we discuss the self-propulsion of spermatozoa through mucus filling the cervical canal, which is taken as a channel with flexible boundaries. The chapter consists of two

parts : in part I, we have taken micropolar fluid model for mucus while in part II simple microfluid model has been considered. The spermatozoa is modelled as a two dimensional sheet which while swimming, sends down lateral waves of finite amplitude along its length. The model also considers the motion of the walls due to mascular activity which is represented by the transverse waves travelling along the flexible walls of the channel in the direction opposite the motion of the sheet. The analysis has been carried out for the inertia free flow under the assumption that the waves travelling along the channel walls and along the sheet are in synchronization under steady state and thus have same wavelength and wave speed.

In our study here, the effect of simple microfluid has been discussed by taking arbitrary and independent variations of the microcontinuum viscosity coefficients, since no experimental data could be ascribed to the simple microfluid parameters. Hence, the results presented here must be used with caution. It has been observed that though the qualitative behaviour of the flow characteristics for a given simple microfluid is similar to those for Newtonian fluid, they show quantitative variations. The results show, reasonably well, the capabilities of microcontinuum approach in the physiological flow problems.

CHAPTER II

FLOW OF SIMPLE MICROFLUID THROUGH AN ARTERY WITH MILD STENOSIS

2.1 INTRODUCTION

It is well known now that flow through stenosed artery can cause serious circulatory disorders. Hence, the flow characteristics of blood through a stenosed artery, have been extensively investigated in recent years. One of the earlier attempts to theoretically analyse the interaction of stenosis with fluid dynamics of blood flow was made by Young (1968) in the case of mild stenosis. This has been subsequently investigated by many other workers, for steady and unsteady cases [Forrester and Young (1970), Lee and Fung (1970), Morgan and Young (1974), Shukla et al. (1979)]. However it may be noted that in these studies blood has been characterised as Newtonian fluid and little attention has been given to the suspension nature of blood. The experimental studies have indicated that, under certain flow conditions, rheology of blood significantly deviates from the Newtonian fluid behaviour due to its suspension nature [Charm & Kurland (1965), Whitmore (1968)]. For example, non parabolic velocity profile, existense of peripheral layer etc., have been observed for the flow through tubes of small diameter [Bugliarello & Sevilla (1970), Cokelet (1972), Goldsmith (1975)]. Thus attempts have been made to rheologically describe blood as a suspension of neutrally buoyant deformable particles in viscous

fluid using microcontinuum approach [Kline et al. (1968), Cowin (1972), Popel et al. (1974), Ariman et al. (1971,1973b), Kang and Eringen (1976)].

In view of this Sinha and Singh (1984), Srivastava (1985) considered couple stress fluid model to study blood flow through stenosed tube. Though their models account for the size effects of the particles, it does not account for the angular velocity and deformability of the suspended particles (Cells). Pralhad and Shultz (1988) have discussed blood flow through a stenosed artery for different diseases by considering polar fluid model for blood, which accounts for the particles' spin. However, the effect of cell deformability, which has pronounced effect on the apparent viscosity, is not considered in this model. Recently, Kang and Eringen (1976) have proposed simple microfluid model for blood. This model accounts for the microspin and micro-stretch of the particles. Therefore in this chapter, an attempt is made to analyse blood flow through a narrow artery with mild stenosis by considering simple microfluid model for blood. The model presented here considers only the localized effects of the stenosis and accounts for cell free layer of blood plasma near the tube wall [Bugliarello and Sevilla (1970), Shukla et al. (1980a,b)]. Further we assume, flow to be steady, as the pulsatile behaviour of blood flow is not very significant in narrow arteries.

2.2 FORMULATION

We consider here two layer-model for blood flow in stenosed artery. The model consists of a central layer of simple microfluid in the core region and a peripheral layer of Newtonian fluid (plasma) near the wall of the tube. The geometrical configuration of stenosis, which is not very well defined, is assumed to be axially symmetric and the radius R(z) (Fig. 2.1) of the stenosed tube is given as follows [Young (1968)]

$$R(z) = \begin{cases} R_o - \dfrac{\delta_s}{2}\left[1 + \cos\dfrac{2\pi}{L_o}\left(z - d - \dfrac{L_o}{2}\right)\right] & d \le z \le L_o+d \\ R_o \qquad \text{otherwise} & \end{cases} \tag{2.1}$$

Here R(z) is the radius of the tube with stenosis, R_o is the constant radius, L_o is the length of the stenosis and δ_s is the maximum height of the stenosis. We further assume that :

(i) Fluids in both the layers are incompressible,

(ii) Body forces and body moments are absent,

(iii) Flow, which is due to the pressure gradient dp/dz, is one-dimensional and axi-symmetric,

(iv) Stenosis is very mild, which implies that the variation of all the fluid quantities (except pressure) along the axial direction is negligible,

v) The length of the tube is large compared to its radius (which on order of magnitude analysis implies that the inertial terms can be neglected),

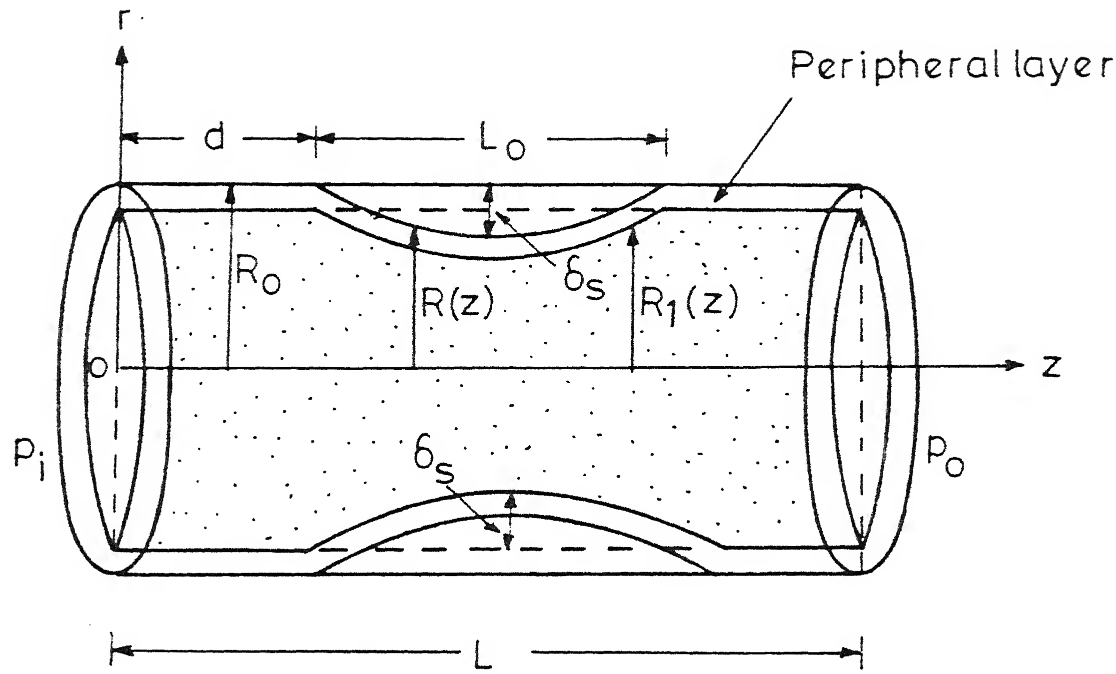


FIG. 2.1 GEOMETRY OF ARTERIAL STENOSIS WITH PERIPHERAL LAYER

(vi) The peripheral layer is of constant thickness, δ. Thus the core region is given by $0 \le r \le R_1(z) = R(z) - \delta$ and peripheral region is given by $R_1(z) \le r \le R(z)$.

Using the above assumption we now write the euqations governing the steady laminar flow in the two regions :

(a) CORE REGION ($0 \le r \le R_1(z)$) : As mentioned above, here we have a simple microfluid flowing under the pressure gradient dp/dz. The field quantitites are, velocity component, $\bar{v} = (0, 0, v_p)$, gyration tensor components ν_{13} and ν_{31} (other components being zero) and pressure, p. Thus the equation of balance of linear momentum as given by equation (1.2) in Chapter I give

$$\frac{\partial}{\partial r}(t_{13}) + \frac{1}{r} t_{13} = \frac{dp}{dz} \equiv P \qquad (2.2)$$

while the equations due to balance of first moment stress (1.3) reduce to

$$\frac{\partial}{\partial r}(\lambda_{113}) + \frac{1}{r}(\lambda_{113} - \lambda_{223}) + t_{31} - s_{31} = 0 \qquad (2.3)$$

$$\frac{\partial}{\partial r}(\lambda_{131}) + \frac{1}{r}(\lambda_{131} - \lambda_{232}) + t_{13} - s_{13} = 0 \qquad (2.4)$$

where t_{ij}, s_{ij}, λ_{ijk} are given by the constitutive equations (1.5 to 1.7) in Chapter I. The non zero components can be written as,

$$t_{13} = 2\mu\, d_{13} + \zeta_1\, \nu_{(13)} + k(\nu_{[13]} - \omega_{13}) \qquad (2.5)$$

$$t_{31} = 2\mu\, d_{31} + \zeta_1\, \nu_{(31)} + k(\nu_{[31]} - \omega_{31}) \tag{2.6}$$

$$s_{13} = 2\mu\, d_{13} + 2\,\zeta_2\, \nu_{(13)} \tag{2.7}$$

$$s_{31} = 2\mu\, d_{31} + 2\,\zeta_2\, \nu_{(31)} \tag{2.8}$$

$$\lambda_{113} = \frac{1}{2}\,(K_1+K_3)\frac{\partial}{\partial r}\,\nu_{31} + \frac{1}{2}\,(K_2+K_4)\frac{\partial}{\partial r}\,\nu_{13} \tag{2.9}$$

$$\lambda_{223} = \left[(K_1+K_3)\nu_{31} + K_2+K_4)\nu_{13})\right] \,/\, 2r \tag{2.10}$$

$$\lambda_{131} = \frac{1}{2}\,(K_1-K_3)\,\frac{\partial\nu_{31}}{\partial r} + \frac{1}{2}\,(K_2-K_4)\,\frac{\partial\nu_{13}}{\partial r} \tag{2.11}$$

$$\lambda_{232} = \left[(K_1-K_3)\,\nu_{31} + (K_2-K_4)\,\nu_{13})\right] \,/\, 2r \tag{2.12}$$

$$d_{13} = \frac{1}{2}\,\left(\frac{\partial v_c}{\partial r}\right) = d_{31} \tag{2.13}$$

$$\omega_{13} = -\,\frac{1}{2}\,\frac{\partial v_c}{\partial r} = -\,\omega_{31}\,, \tag{2.14}$$

$$\nu_{(13)} = \frac{\nu_{13}+\nu_{31}}{2} = \nu_{(31)} \text{ and } \nu_{[13]} = \frac{\nu_{13}-\nu_{31}}{2} = -\nu_{[31]} \tag{2.15}$$

μ, ζ_1, ζ_2, k, γ_1 to γ_{15} are the viscosity co-efficients. ζ_1, ζ_2 and k have the dimensions of viscosity μ, while γ_i's are of the dimensions μL^2. These co-efficients satisfy the following restrictions [Kang and Eringen (1976)],

$$\mu \geq 0 \ , \quad 2\zeta_2 - \zeta_1 \geq k \geq 0, \ 4\mu(\zeta_2 - \frac{1}{2}\zeta_1)^2 - (\zeta_1/2)^2 \geq 0$$

$$\gamma_1 + \gamma_2 + \ldots\ldots + \gamma_{15} \geq 0, \quad \gamma_2 + \gamma_{11} + \gamma_{14} \geq 0 \ . \tag{2.16}$$

K_1, K_2, K_3 and K_4 are as defined in chapter I.

Here $\nu_{(13)}$ and $\nu_{[13]}$ give the symmetric and skew symmetric parts of the ν_{13}. The symmetric part i.e. $\nu_{(13)}$ represents local stretchings and shearings, while the skew symmetric part, $\nu_{[13]}$ give local intrinsic angular velocity of the substructure.

(b) PERIPHERAL REGION ($R_1(z) \leq r \leq R(z)$) : The equation for the Newtonian fluid (plasma) flow with $\bar{v} = (0, 0, v_p)$ and constant viscosity μ under pressure gradient $\frac{dp}{dz}$ is,

$$\mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_p}{\partial r} \right) \right] = P \ . \tag{2.17}$$

THE BOUNDARY CONDITIONS

(i) No slip condition gives,

$$v_p = 0 \quad \text{at} \quad \overset{r}{\mu} = R(z) \ . \tag{2.18}$$

(ii) Symmetry of the flow implies that,

$$\frac{\partial v_c}{\partial r} = 0, \quad \nu_{(13)} = 0 \quad \nu_{[13]} = 0 \quad \text{at } r = 0 \ . \tag{2.19}$$

(iii) Continuity of velocity at the interface implies,

$$v_c(R_1) = v_p(R_1) . \tag{2.20}$$

(iv) Continuity of shear stress at the interface gives

$$\mu \frac{\partial v_p}{\partial r}\bigg|_{r=R_1} = t_{13}\bigg|_{r=R_1} . \tag{2.21}$$

(v) The determination of proper boundary conditions for $\nu_{(kl)}$ and $\nu_{[kl]}$ at the interface is rather difficult.

Here, following Kang and Eringen (1976), we assume similar condition for the microstretching and microspin at the interface. Thus, at $r = R_1(z)$, we have

$$\nu_{(13)} = A_1 \frac{\partial v_c}{\partial r} \tag{2.22}$$

$$\nu_{[13]} = A_2 \frac{\partial v_c}{\partial r} \tag{2.23}$$

where A_1 and A_2 are constants.

2.3 ANALYSIS

Integrating the equation (2.2) using the equation (2.5) and the boundary condition (2.19) at $r = 0$, we obtain

$$t_{13} = (\mu + \frac{k}{2}) \frac{\partial v_c}{\partial r} + \zeta_1 \nu_{(13)} + k\, \nu_{[13]} = \frac{rP}{2} \tag{2.24}$$

which gives the interface condition (2.21) for stress in the following form :

$$\mu \left.\frac{\partial v_p}{\partial r}\right|_{r=R_1} = \left. t_{13} \right|_{r=R_1} = \frac{R_1 P}{2} . \tag{2.25}$$

Equation (2.17) alongwith the boundary conditions (2.18) and (2.25) gives v_p as,

$$v_p = -\frac{P}{4\mu}(R^2 - r^2) \qquad R_1 \le r \le R . \tag{2.26}$$

Now substituting for λ_{113}, λ_{223}, t_{31}, s_{31}, λ_{131}, λ_{232}, t_{13} and s_{13} from eqns. (2.5) to (2.12) in eqns. (2.3) and (2.4), we get

$$(K_1+K_3)\,\nabla^2 \nu_{31} + (K_2+K_4)\,\nabla^2 \nu_{13} + 2(\zeta_1 - 2\zeta_2)\,\nu_{(31)} + 2k\left(\nu_{[31]} - \frac{1}{2}\frac{\partial v_c}{\partial r}\right) = 0 \tag{2.27}$$

$$(K_1-K_3)\,\nabla^2 \nu_{31} + (K_2-K_4)\,\nabla^2 \nu_{13} + 2(\zeta_1 - 2\zeta_2)\,\nu_{(13)} + 2k\left(\nu_{[13]} + \frac{1}{2}\frac{\partial v_c}{\partial r}\right) = 0 \tag{2.28}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}$

On adding (2.27) and (2.28) we get

$$K_1\,\nabla^2 \nu_{31} + K_2\,\nabla^2 \nu_{13} + (\zeta_1 - 2\zeta_2)\,[\nu_{13} + \nu_{31}] = 0 \tag{2.29}$$

while subtraction of (2.28) from (2.27) gives

$$K_3 \nabla^2 \nu_{31} + K_4 \nu_{13} - k [\nu_{13} - \nu_{31} + \frac{\partial v_c}{\partial r}] = 0 . \tag{2.30}$$

Using eqn.(2.24), the eqns. (2.29) and (2.30) can be rewritten as follows :

$$K_1[\nabla^2 - \alpha_1^2] \nu_{31} + K_2 [\nabla^2 - \alpha_2^2] \nu_{13} = 0 \tag{2.31}$$

$$K_3 [\nabla^2 - \alpha_3^2] \nu_{31} + K_4 [\nabla^2 - \alpha_4^2] \nu_{13} = \frac{krP}{(2\mu + k)} \tag{2.32}$$

where

$$\left.\begin{aligned} K_1 \alpha_1^2 &= - (\zeta_1 - 2\zeta_2) = K_2 \alpha_2^2 \\ K_3 \alpha_3^2 &= - k (2\mu + \zeta_1)/(2\mu + k) \\ K_4 \alpha_4^2 &= k (2\mu - \zeta_1)/(2\mu + k) \end{aligned}\right\} \tag{2.33}$$

The equations (2.31) and (2.32) suggest the following forms for ν_{13} and ν_{31}

$$\nu_{13} = b_1 f_1 I_1 (\alpha r) + b_2 f_2 I_1 (\beta r) + \frac{P}{4\mu} r \tag{2.34}$$

$$\nu_{31} = b_1 I_1 (\alpha r) + b_2 I_1 (\beta r) - \frac{P}{4\mu} r \tag{2.35}$$

where

$$\left.\begin{aligned} f_1 &= -\overline{K}_4 (\overline{\alpha}^2 - \overline{\alpha}_4^2)/\overline{K}_3(\overline{\alpha}^2 - \overline{\alpha}_3^2) \\ f_2 &= -\overline{K}_2 (\overline{\beta}^2 - \overline{\alpha}_2^2)/\overline{K}_1(\overline{\beta}^2 - \overline{\alpha}_1^2) \end{aligned}\right\} \tag{2.36}$$

$$\bar{\alpha}_i = \alpha_i R_o, \quad \bar{\alpha} = \alpha R_o, \quad \bar{\beta} = \beta R_o, \quad \bar{K}_i = \frac{K_i}{\mu R_o^2} \qquad (2.37)$$

and α^2 and β^2 are obtained using the equations (2.31) and (2.32) and are given as the positive roots of the quadratic equation

$$x^2 (K_2K_3 - K_1K_4) + x[(\alpha_2^2 + \alpha_3^2) K_2K_3 - (\alpha_1^2 + \alpha_4^2) K_1K_4]$$

$$+ \alpha_2^2 \alpha_3^2 K_2K_3 - \alpha_1^2 \alpha_4^2 K_1K_4 = 0 \qquad (2.38)$$

[$I_1(x)$ is the modified Bessel function of order 1].

Substituting the expressions for ν_{13} and ν_{31} in eqn. (2.24) and integrating we get,

$$v_c = b_3 + \frac{2}{2\mu+k} \left\{ \frac{Pr}{4}^2 - \frac{\zeta_1}{2} \left[\frac{b_1}{2} (1 + f_1) I_o(\alpha r) \right.\right.$$

$$\left. + \frac{b_2}{2} (1 + f_2) I_o(\beta r) \right] - \frac{k}{2} \left[\frac{b_1}{2} (1 - f_1) I_o(\alpha r) \right.$$

$$\left.\left. + \frac{b_2}{2} (1 - f_2) I_o(\beta r) - \frac{Pr^2}{4\mu} \right] \right\} \qquad (2.39)$$

where b_3 is constant of integration.

The constants b_1, b_2 and b_3 are determined using conditions (2.19), (2.21) and (2.22) along with the equations (2.23) and (2.25). Thus, the field quantities $\nu_{(13)}$ (micro-stretchings), $\nu_{[13]}$ (micro-spins) and the velocity v_c are obtained in the following form :

$$\nu_{(13)} = -\frac{PR_o}{4\mu}\left[b_1(1+f_1)\ I_1(\bar{\alpha}\bar{r}) + b_2(1+f_2)\ I_1(\bar{\beta}\bar{r})\right] \qquad (2.40)$$

$$\nu_{[13]} = -\frac{PR_o}{4\mu}\left[b_1(1-f_1)\ I_1(\bar{\alpha}\bar{r}) + b_2(1-f_2)\ I_1(\bar{\beta}\bar{r}) - \bar{r}\right] \qquad (2.41)$$

$$v_c = \frac{PR_o^2}{4\mu}(\bar{r}^2-\bar{R}^2) - \frac{PR_o\ R_1}{4\mu}\left[d_1\left\{I_o(\bar{\alpha}\bar{R}_1) - I_o((\bar{\alpha}\bar{r})\right\}\right.$$

$$\left. + d_2\left\{I_o(\bar{\beta}\bar{R}_1) - I_o((\bar{\beta}\bar{r})\right\}\right] \qquad (2.42)$$

Here,

$$\left.\begin{aligned}
d_i &= \frac{-b_i\ e_i}{a_i I_1(a_i\bar{R}_1)} \\
b_1 &= \left[(1+2A_2)g_2 - 2A_1h_2\right] / \left[g_1h_2 - g_2h_1\right] \\
b_2 &= \left[(1+2A_2)g_1 - 2A_1h_1\right] / \left[g_1h_2 - g_2h_1\right] \\
g_i &= (-1)^{i+1}\left\{\frac{(1+f_i)}{2} - A_1e_i\right\} \\
h_i &= (-1)^{i+1}\left\{\frac{(1-f_i)}{2} - A_2e_i\right\} \\
f_i &= \frac{(a_i^2 - \bar{\alpha}_4^2)\ \bar{K}_1\bar{K}_4 + (a_i^2 - \bar{\alpha}_2^2)\ \bar{K}_2\bar{K}_3}{\bar{K}_1\bar{K}_3\ (\bar{\alpha}_1^2 - \bar{\alpha}_3^2)} \\
e_i &= \frac{(E_1-E_3) + (E_1+E_3)\ f_i}{(2+E_3)}
\end{aligned}\right\} \qquad (2.43)$$

$$a_1 = \bar{\alpha} \; ; \; a_2 = \bar{\beta} \; ; \; E_1 = -\frac{\zeta_1}{\mu} \; ; \; E_2 = \frac{\zeta_2}{\mu} \; ; \; E_3 = \frac{k}{\mu} \; ; \; \bar{r} = r/R_o$$

$$\bar{R}_1 = \frac{R_1(z)}{R_o} \; ; \qquad \bar{R} = R(z)/R_o = 1 - \frac{\bar{\delta}_s}{2}\left\{1+\cos 2\pi \left(\bar{z}-\bar{d}-\frac{\bar{L}_o}{2}\right)\right\},$$

$\bar{d} = d/L$ and $\bar{L}_o = L_o/L$.

It may be remarked that $\delta=0$ i.e. $R_1(z) = R(z)$ gives the case of no peripheral layer and the tube is full of simple microfluid, while $R_1(z) = 0$, gives no core region of simple microfluid.

The flow flux Q which is defined as,

$$Q = 2\pi \int_o^{R_1} r\, v_c\, dr + 2\pi \int_{R_1}^{R} r\, v_p\, dr$$

can be obtained in the following form using equations (2.25) and equation (2.42)

$$Q = \frac{\pi R_o^4}{8\mu} \frac{dp}{dz} F\,[R(z),\, R_1(z)] \tag{2.44}$$

where

$$\bar{F}(z) = \bar{F}[\bar{R}(z),\, \bar{R}_1(z)] = -\bar{R}^4 - \frac{2\bar{R}_1^3}{(2+E_3)}\left[d_1\left\{I_o(\bar{\alpha}\bar{R}_1) - \frac{2I_1(\bar{\alpha}\bar{R}_1)}{\bar{\alpha}\bar{R}_1}\right\}\right.$$

$$\left. + d_2\left\{I_o(\bar{\beta}\bar{R}_1) - \frac{2I_1(\bar{\beta}\bar{R}_1)}{\bar{\beta}\bar{R}_1}\right\}\right]. \tag{2.45}$$

Using equation (2.44) alongwith the conditions

$$p = p_i \quad \text{at} \quad z = 0 \quad \& \quad p = p_o \quad \text{at} \quad z = L ,$$

we get,

$$p_o - p_i = \frac{8\mu Q}{\pi R_o^4} \int_o^L \frac{dz}{\bar{F}(z)} \tag{2.46}$$

It may be noted that $\bar{F} = \bar{F}(R_o, R_o - \delta)$ is constant in the region $o \le z \le d$, $d+L_o \le z \le L$.

Thus,

$$p_o - p_i = \frac{8\mu Q}{\pi R_o^4} \left[\frac{L-L_o}{\bar{F}(R=R_o)} + \int_d^{d+L_o} \frac{dz}{\bar{F}(z)} \right] . \tag{2.47}$$

The peripheral resistance λ, is given in the non-dimensional form as [Young (1968)],

$$\bar{\lambda} = \frac{\lambda}{\lambda_o} = \left[\frac{1-\bar{L}_o}{\bar{F}(\bar{R}=1)} + \int_{\bar{d}}^{\bar{d}+\bar{L}_o} \frac{d\bar{z}}{\bar{F}(z)} \right] \tag{2.48}$$

where

$$\lambda = (p_i - p_o)/Q , \quad \lambda_o = 8\mu L/(\pi R_o^4) \tag{2.49}$$

The equation (2.48) under the transformation

$$\phi = \frac{1}{2} - \frac{1}{\bar{L}_o} (\bar{z} - \bar{d} - \frac{\bar{L}_o}{2})$$

can be written as

$$\bar{\lambda} = \left[\frac{1-\bar{L}_o}{\bar{F}(\bar{R}=1)} + \bar{L}_o \int_o^1 \frac{d\phi}{\bar{F}(\phi)} \right] . \tag{2.50}$$

The shearing stress at the wall τ_w in nondimensional form is given by

$$\bar{\tau}_w = \frac{\pi R_o^3}{4\mu Q} \tau_w = - \frac{\bar{R}}{\bar{F}} \tag{2.51}$$

2.4 RESULTS AND DISCUSSION

The resistance to the flow, λ and the wall shear stress, τ_w are the two important characteristics of blood flow through stenosed artery. Therefore in the following we discuss the effect of various parameters on these flow characteristics. $\bar{\lambda}$ and $\bar{\tau}_w$, given by eqns. (2.50) and (2.51) respectively, are the non-dimensional forms of λ and τ_w. The integral appearing in eqn. (2.50) is not amenable to analytical form, so we numerically integrate it using Simpson's $\frac{1}{3}$ rule. Polynomial approximations are taken for the Bessel functions $I_o(x)$, $I_1(x)$. Apart from the usual dimensionless parameters, i.e. $\bar{\delta}_s$ (the maximum height of the stenosis), $\bar{L}_o$ (the length of the stenosis) and $\bar{\delta}$ (the peripheraql layer thickness), we encounter simple microfluid parametrs (E_1, E_2, E_3, K_1, K_2, K_3, K_4) in the expressions of $\bar{\lambda}$and $\bar{\tau}_w$.

The parameters E_1, E_2 and E_3 correspond to the deformability and rigidity of the particles, e.g. at fixed μ higher values of k represent more rigid structure, while higher values of $|\zeta_1|$ and $|\zeta_2|$ represent more flexible structure. However the precise rates at which these parameters increase or decrease with deformability or rigidity are not known. Further, these constitutive co-efficients depend upon the initial concentration of the particles as well as on the particle shape, deformability, and the fluid viscosity; but again their explicit relations are not known. In view of this, we study the effect of independent variation of these parameters on the flow characteristics. The parameters K_1, K_2, K_3 and K_4 are the linear combinations of the viscosity co-efficients γ_1 to γ_{15} and have the dimensions of μL^2. Thus the non-dimensional parameters $\bar{K}_i$ (= $K_i/\mu L^2$) for a narrow artery have been chosen as $\bar{K}_1 = 1.1$ $\bar{K}_2 = 1.0$, $\bar{K}_3 = -0.9$, $\bar{K}_4 = 0.6$ [Kang and Eringen (1976)].

In table I and II we discuss the case when $A_1 = A_2 = 0$ i.e. the microspin and the microstretch at the interface are zero while figs. (2.2 to 2.11) are for $A_1 = A_2 = 0.25$. It is clear from Table 1 and 2 that both $\bar{\lambda}$ and $\bar{\tau}_w$ (calculated at the maximum height of the stenosis) increase as $\bar{\delta}_s$ increases, but decrease as the peripheral layer thickness increases. This is similar to the behaviour of these characteristics as observed by Shukla et al. (1979) who took Newtonian fluids in both the regions but with a different viscosity. The resistance to the flow gets enhanced with the increase in E_1 and E_3 but with decrease in E_2.

Fig. (2.2 to 2.4) show the variation of $\bar{\lambda}$ vs $\bar{\delta}_s$ when $\bar{\delta}$ (peripheral layer thickness) = 0.05. It is observed that in the case of simple microfluid also, the resistance to the flow becomes more as $\bar{\delta}_s$ (height of the stenosis) or $\bar{L}_o$ (length of the stenosis) increases.

Fig.(2.5) shows the effect of the increase in peripheral layer thickness for a given simple microfluid. It is clear that increase in peripheral layer thickness decreases $\bar{\lambda}$. Further $\bar{\lambda}$ is more when the tube is filled with simple microfluid only in comparison to the case of a Newtonian fluid filled tube. Thus the presence of peripheral layer helps blood flow in diseased cases.

The effect of the simple microfluid parameters E_1 and E_2 are opposite to each other. It is clear from the figures that an increase in E_1 enhances the resistance to the flow while that in E_2 reduces it. Unlike the case of $A_1 = A_2 = 0$ where the effect of E_1 and E_2 are not significant, here $\bar{\lambda}$ shows appreciable variations with E_1 and E_2. The effect of the parameter E_3, in this case, is to reduce $\bar{\lambda}$, as E_3 varies between 0.1 to 0.2 (refer Figs 2.3 and 2.4), however $\bar{\lambda}$ increases for higher values of E_3 (Figs. 2.3 ad 2.4). The combined effects of $\bar{L}_o$ and $\bar{\delta}_s$ with the simple microfluid parameters are further elaborated in figs. (2.6 to 2.8).

In figs. (2.9 to 2.11) the shear stress $\bar{\tau}_w$ at the maximum height of the stenosis, is plotted for various values of E_1, E_2

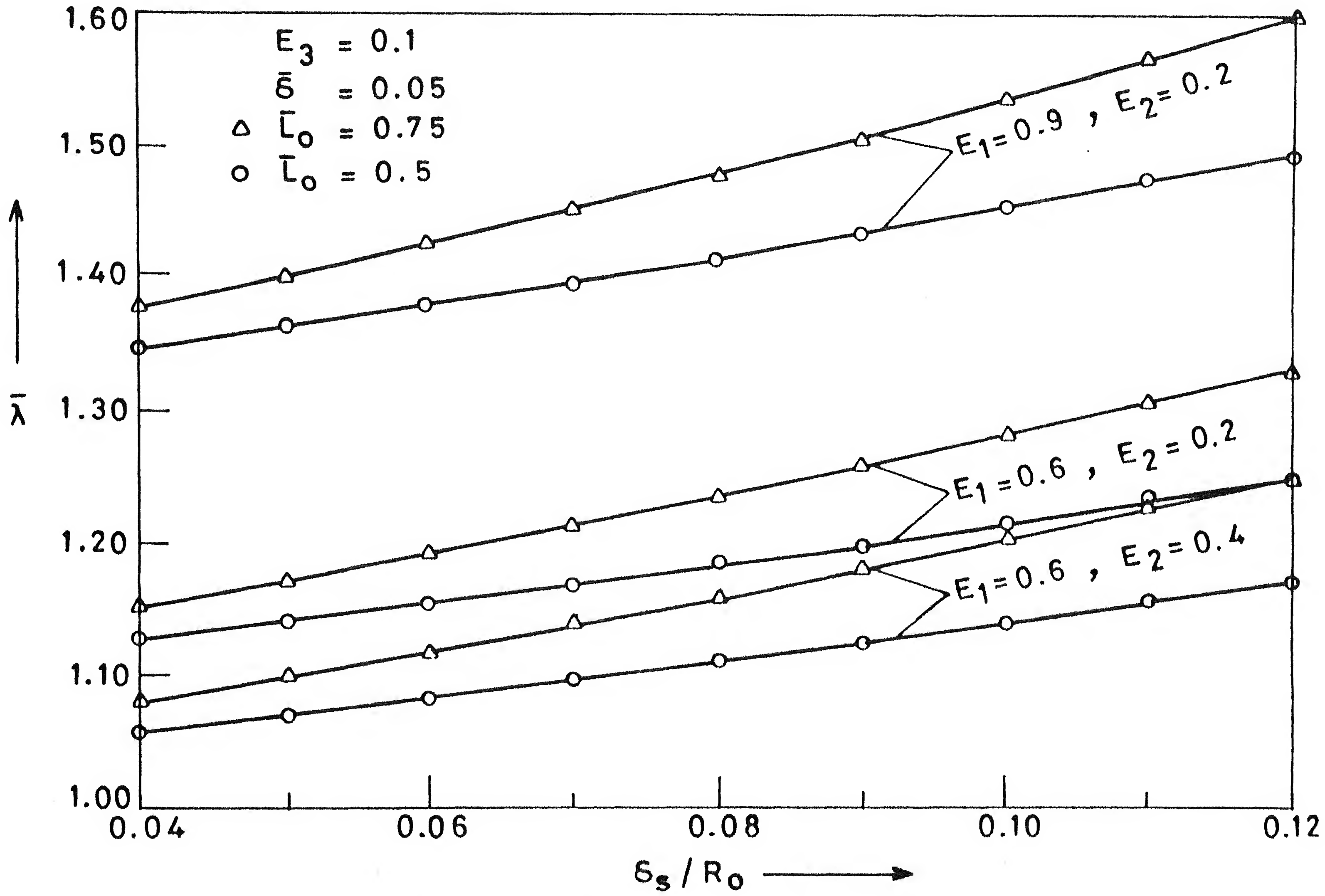


FIG.2.2 VARIATION OF $\bar{\lambda}$ WITH δ_s/R_o FOR DIFFERENT L_o/L E_1 AND E_2.

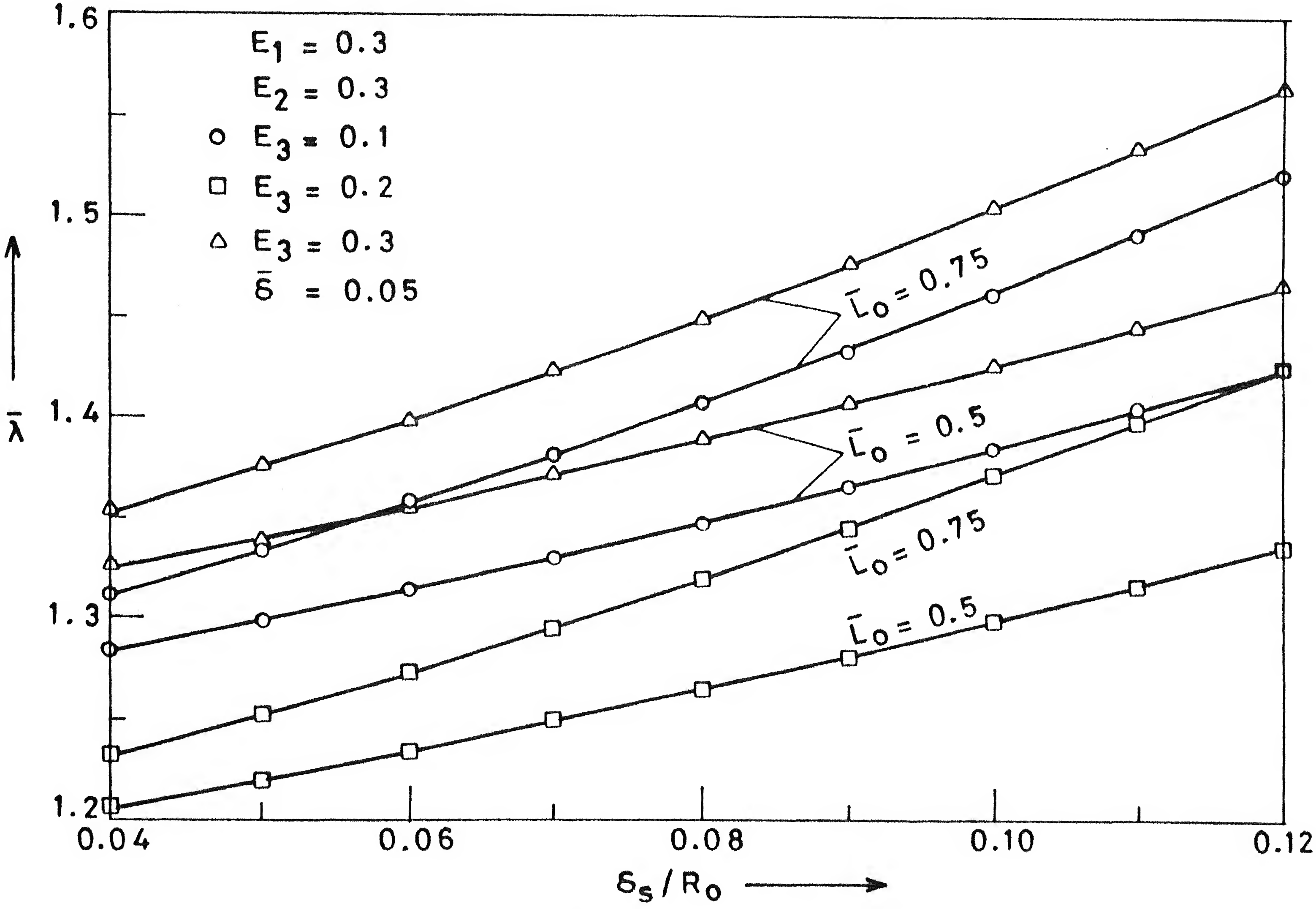


FIG. 2.3 VARIATION OF $\bar{\lambda}$ WITH δ_s/R_o FOR DIFFERENT L_o/L & E_3.

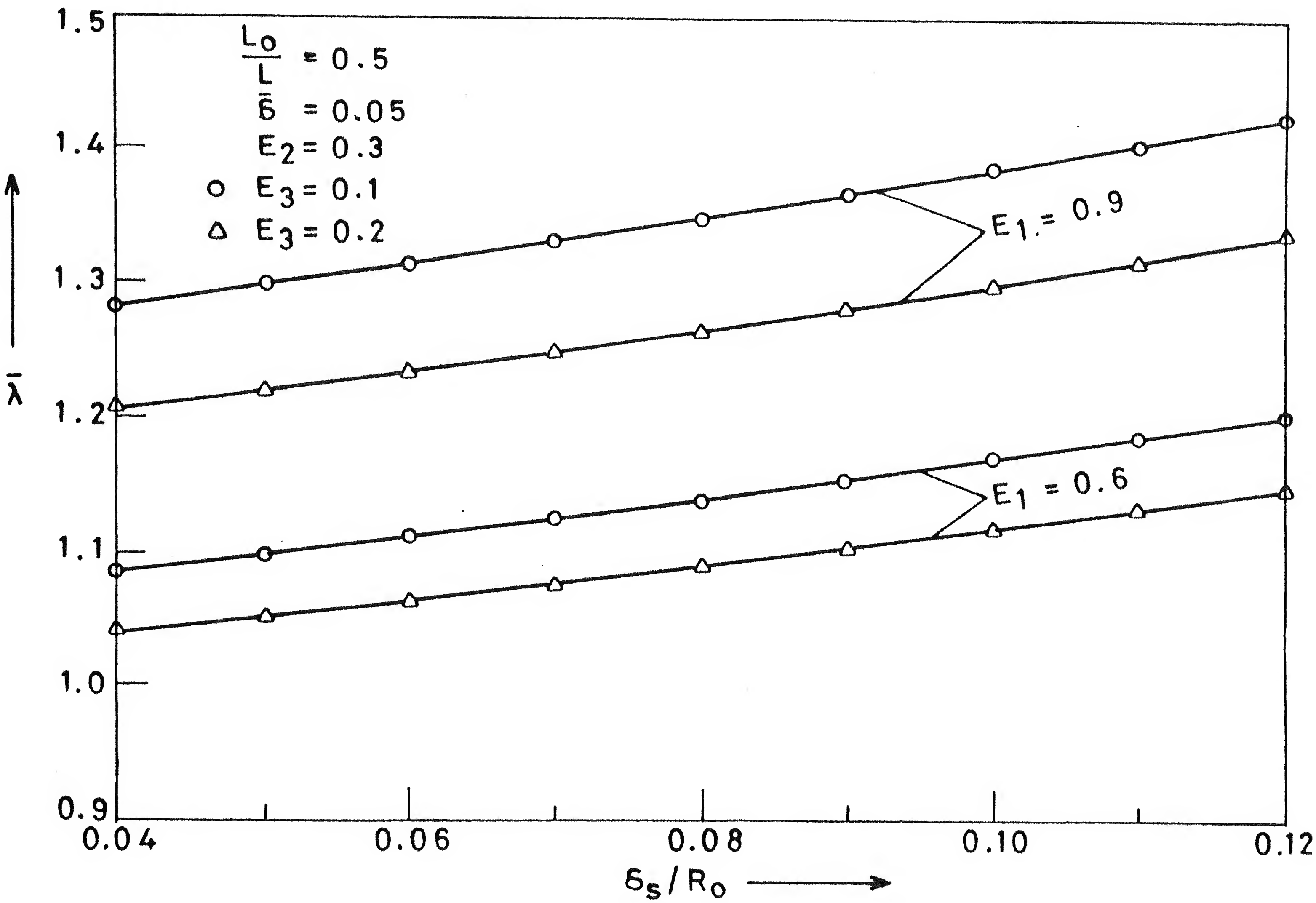


FIG.2.4 VARIATION OF $\bar{\lambda}$ WITH δ_s / R_o FOR DIFFERENT E_1 & E_3.

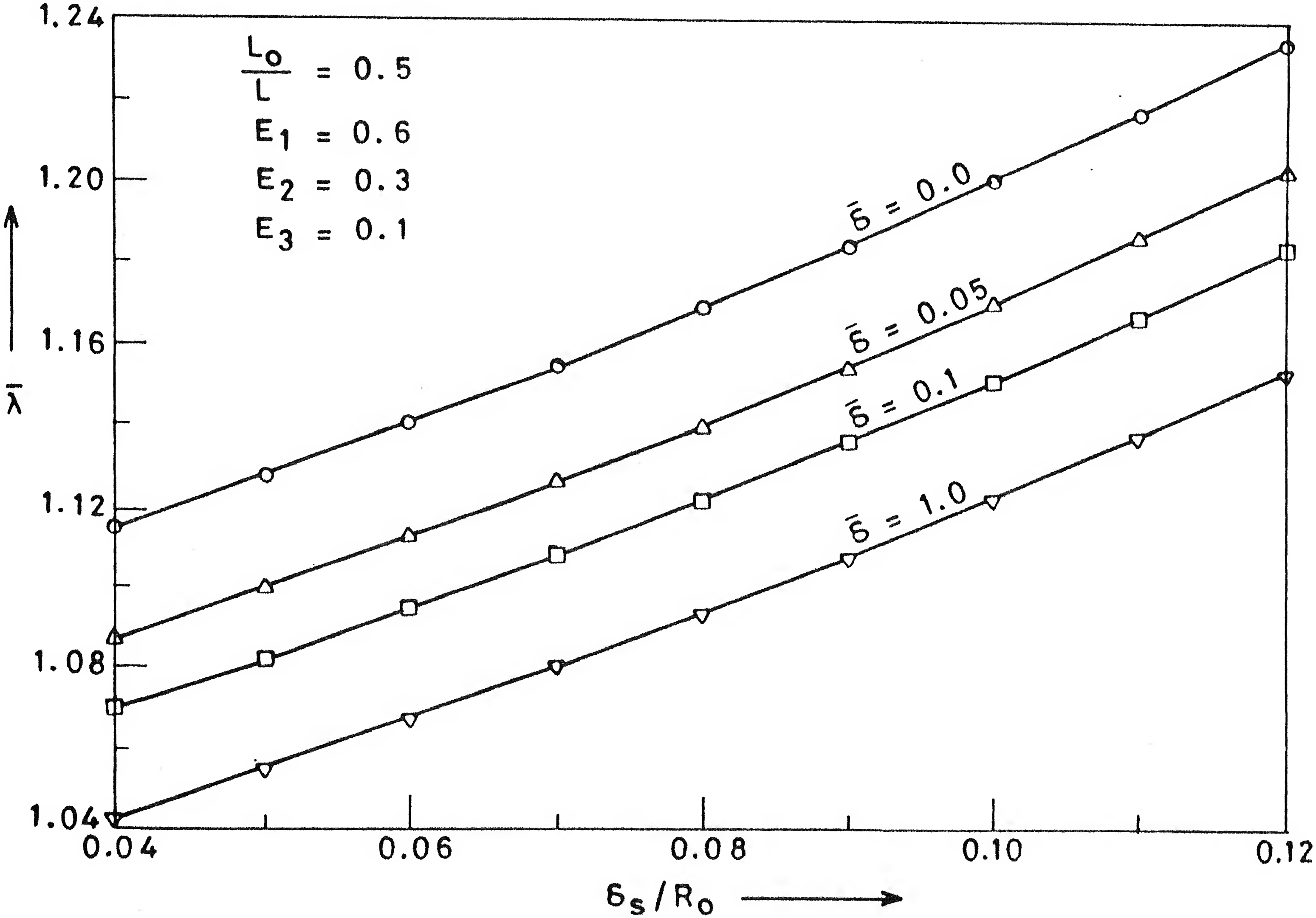


FIG.2.5 VARIATION OF $\bar{\lambda}$ WITH δ_s/R_o FOR DIFFERENT $\bar{\delta}$ (PERIPHERAL LAYER WIDTH)

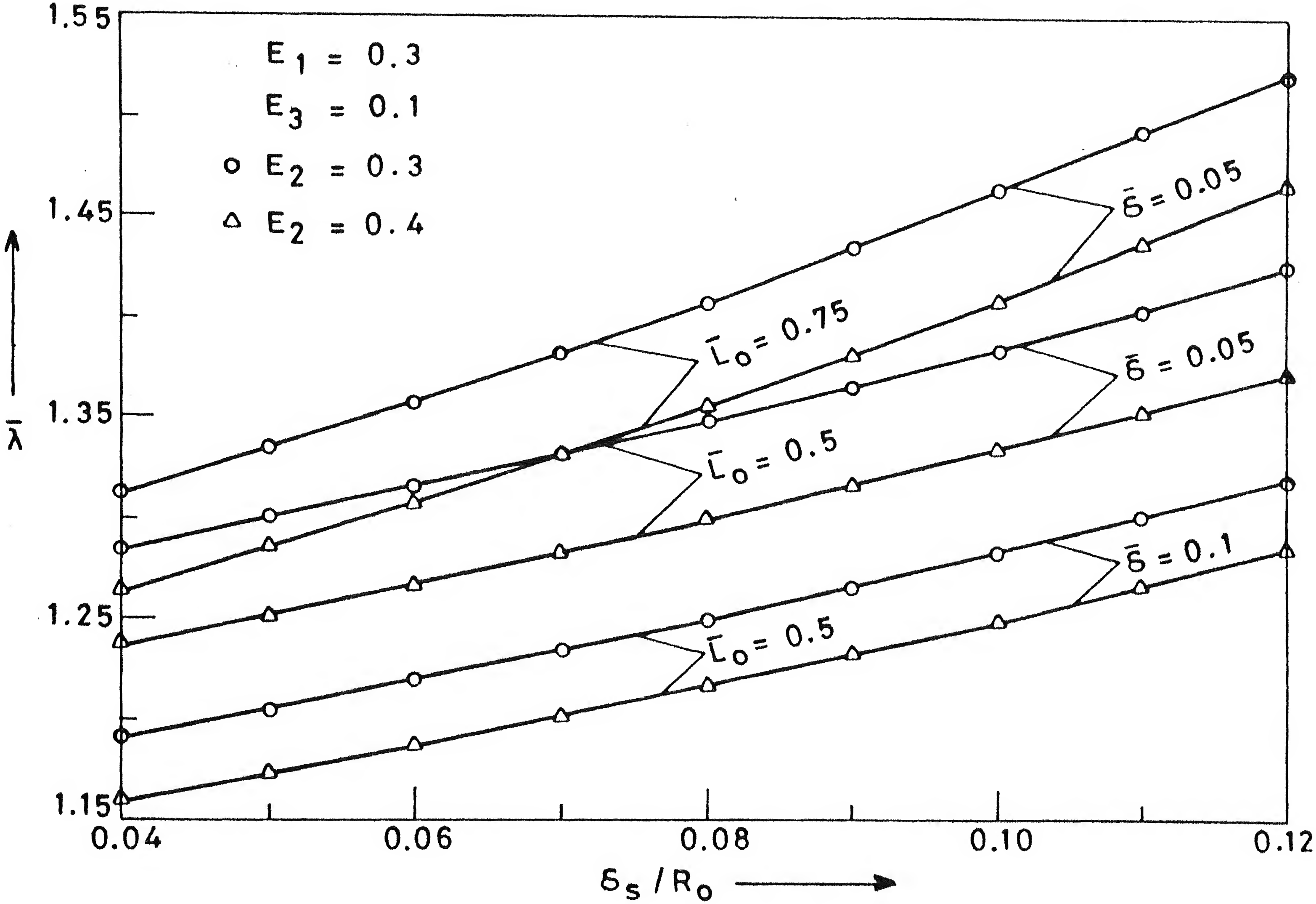


FIG.2.6 VARIATION OF $\bar{\lambda}$ WITH δ_s / R_o FOR DIFFERENT E_2 L_o/L AND $\bar{\delta}$.

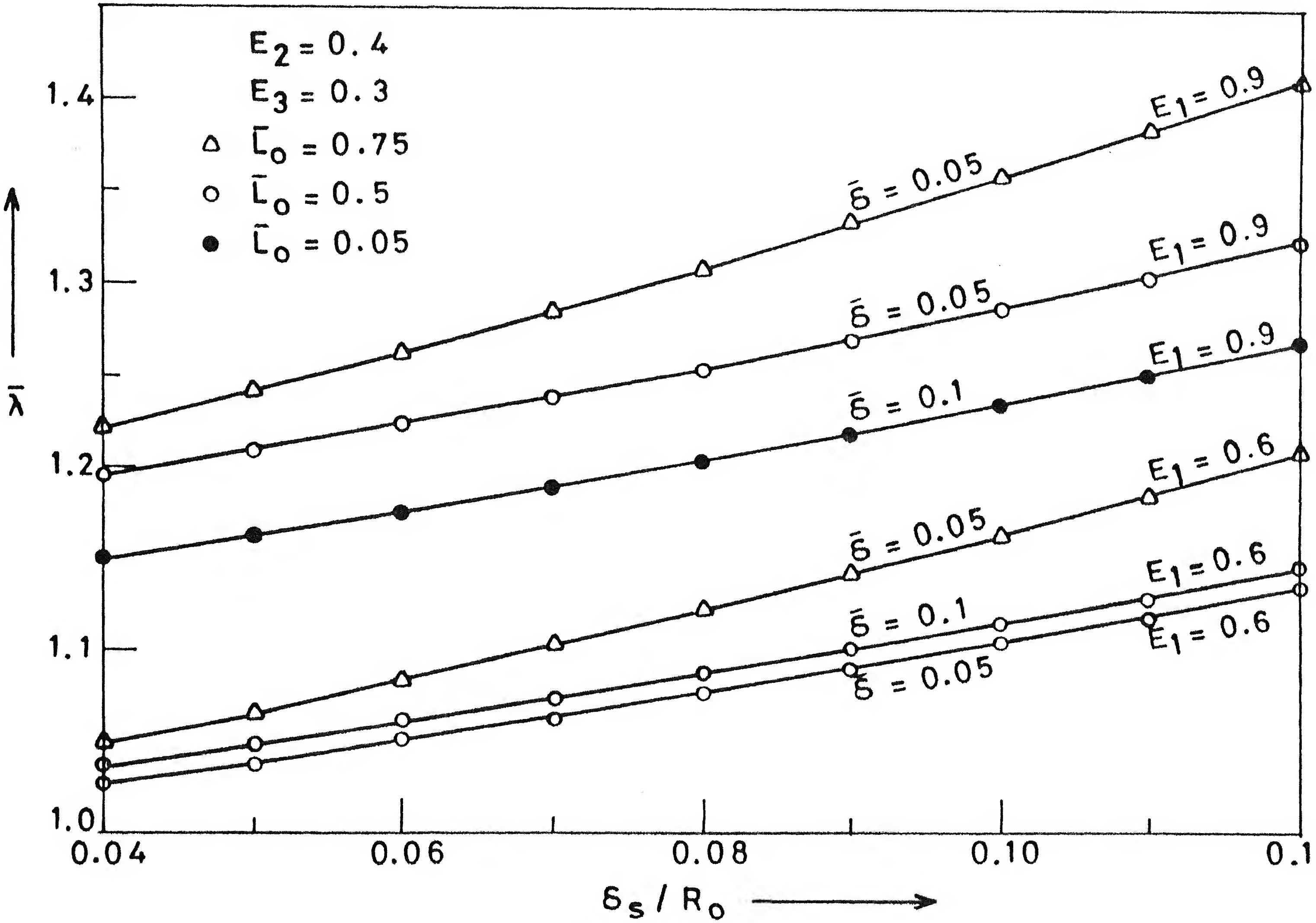


FIG.2.7 VARIATION OF $\bar{\lambda}$ WITH δ_s / R_o FOR DIFFERENT E_1, L_o/L AND $\bar{\delta}$.

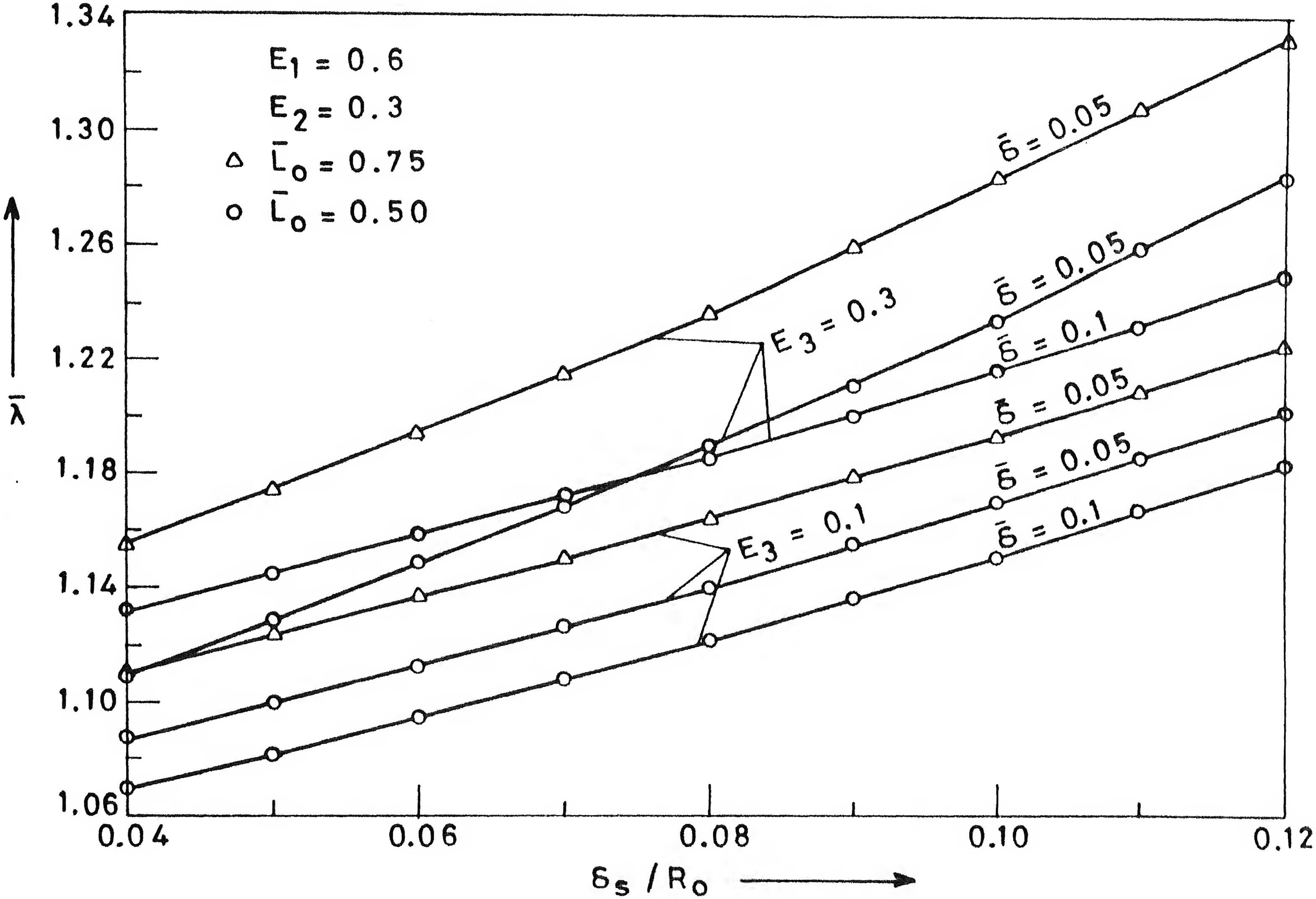


FIG.2.8 VARIATION OF $\bar{\lambda}$ WITH δ_s/R_o FOR DIFFERENT L_o/L, $\bar{\delta}$ AND E_3.

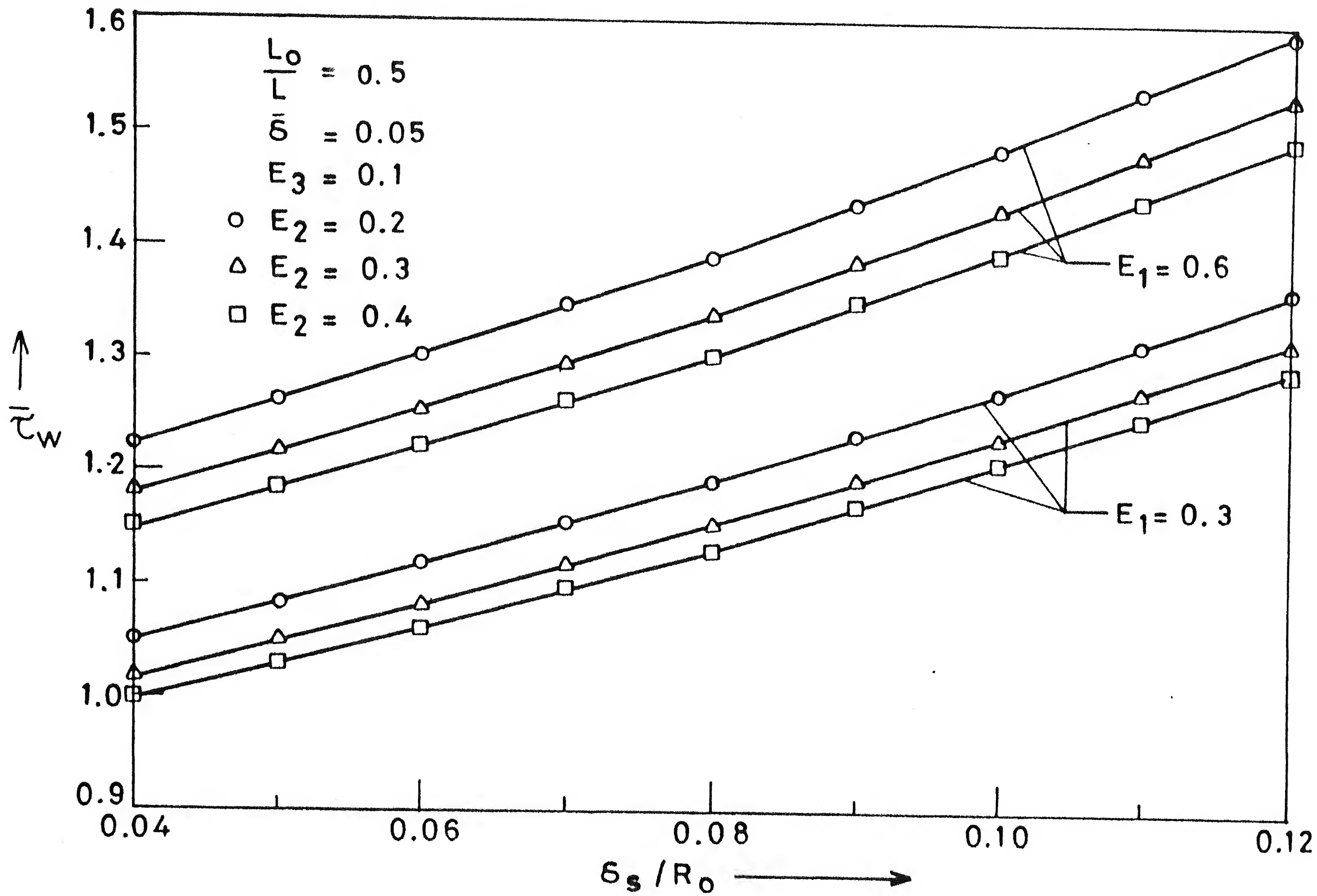


FIG. 2.9 VARIATION OF $\bar{\tau}_w$ WITH δ_s / R_o FOR DIFFERENT E_1 & E_2.

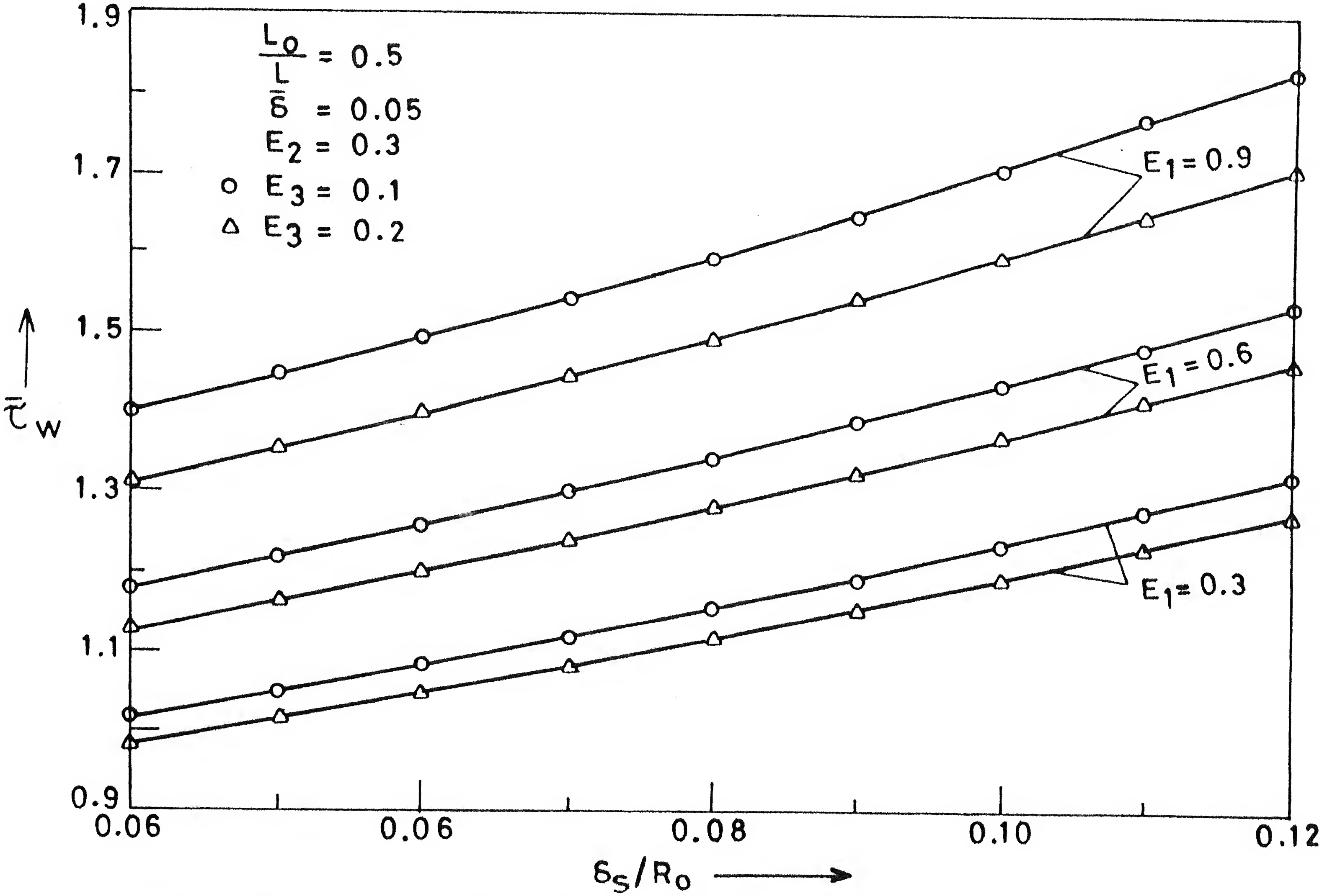


FIG.2.10 VARIATION OF $\bar{\tau}_w$ WITH δ_s / R_o FOR DIFFERENT E_1 & E_3.

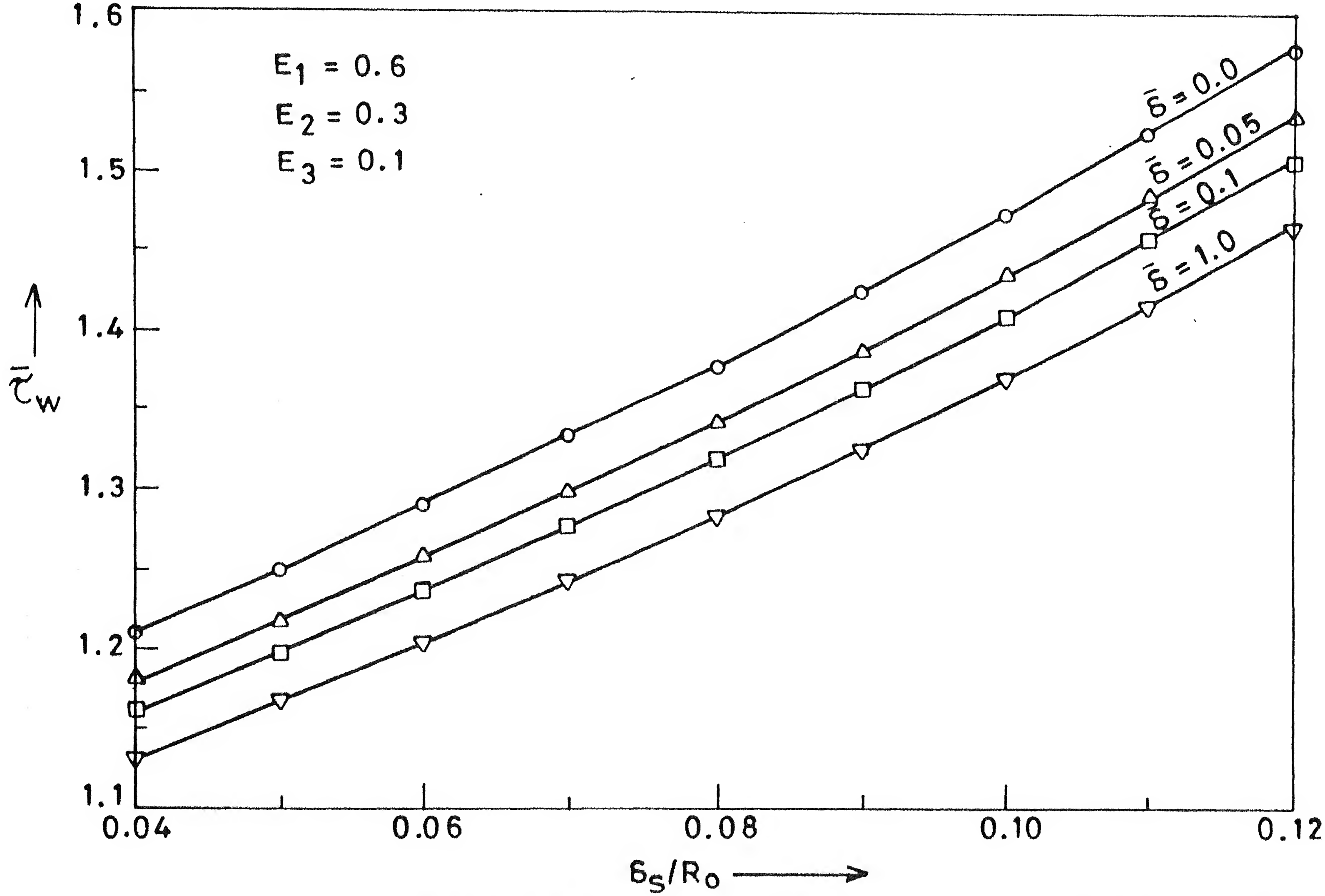


FIG.2.11 VARIATION OF $\bar{\tau}_w$ WITH δ_s/R_o FOR DIFFERENT $\bar{\delta}$ (PERIPHERAL LAYER WIDTH).

and E_3. The qualitative effect of these parameters on $\bar{\tau}_w$ is similar to what had been observed in the case of $\bar{\lambda}$. It is noted from fig. (2.11) that $\bar{\tau}_w$ increases as $\bar{\delta}_s$ increases and this effect is further enhanced in the case of simple microfluid.

Table I

PERIPHERAL LAYER THICKNESS ($\bar{\delta}$) = 0.1						
	$\bar{\delta}_s$	E_1	E_2	E_3	$\bar{\lambda}$	$\bar{\tau}_w$
1.	0.04	0.5	0.3	0.07	1.1386791	1.2284889
2.	0.04	0.5	0.3	0.09	1.1704005	1.2609274
3.	0.04	0.5	0.5	0.07	1.386752	1.2284844
4.	0.04	0.5	0.5	0.09	1.1703953	1.2609215
5.	0.04	1.0	0.3	0.07	1.1388472	1.2286735
6.	0.04	1.0	0.3	0.09	1.1706287	1.2611773
7.	0.04	1.0	0.5	0.07	1.1388398	1.2286652
8.	0.04	1.0	0.5	0.09	1.706185	1.2611660
9.	0.08	0.5	0.3	0.07	1.1917793	1.3878380
10.	0.08	0.5	0.3	0.09	1.2242435	1.4217867
11.	0.08	0.5	0.5	0.07	1.1917751	1.3878330
12.	0.08	0.5	0.5	0.09	1.2242379	1.4217803
13.	0.08	1.0	0.3	0.07	1.1919561	1.3880483
14.	0.08	1.0	0.3	0.09	1.2244831	1.4220702
15.	0.08	1.0	0.5	0.07	1.1919482	1.3880389
16.	0.08	1.0	0.5	0.09	1.2244724	1.4220572

Table II

	PERIPHERAL LAYER THICKNESS ($\bar{\delta}$) = 0.2					
	$\bar{\delta}_s$	E_1	E_2	E_3	$\bar{\lambda}$	$\bar{\tau}_w$
1.	0.04	0.5	0.3	0.07	1.0936822	1.1811299
2.	0.04	0.5	0.3	0.09	1.1095525	1.1970510
3.	0.04	0.5	0.5	0.07	1.0936797	1.1811270
4.	0.04	0.5	0.5	0.09	1.1095492	1.1970474
5.	0.04	1.0	0.3	0.07	1.0937872	1.1812405
6.	0.04	1.0	0.3	0.09	1.1096913	1.1971972
7.	0.04	0.5	0.3	0.07	1.0937823	1.1812353
8.	0.04	1.0	0.5	0.09	1.1096849	1.1971902
9.	0.08	0.5	0.3	0.07	1.1451915	1.3362423
10.	0.08	0.5	0.3	0.09	1.1613095	1.3524337
11.	0.08	0.5	0.5	0.07	1.1451889	1.3362394
12.	0.08	0.5	0.5	0.09	1.1613060	1.3524296
13.	0.08	1.0	0.3	0.07	1.1453001	1.3363623
14.	0.08	1.0	0.3	0.09	1.1614529	1.3525913
15.	0.08	1.0	0.5	0.07	1.1452950	1.3363567
16.	0.08	1.0	0.5	0.09	1.1614462	1.3525839

2.5 CONCLUSION

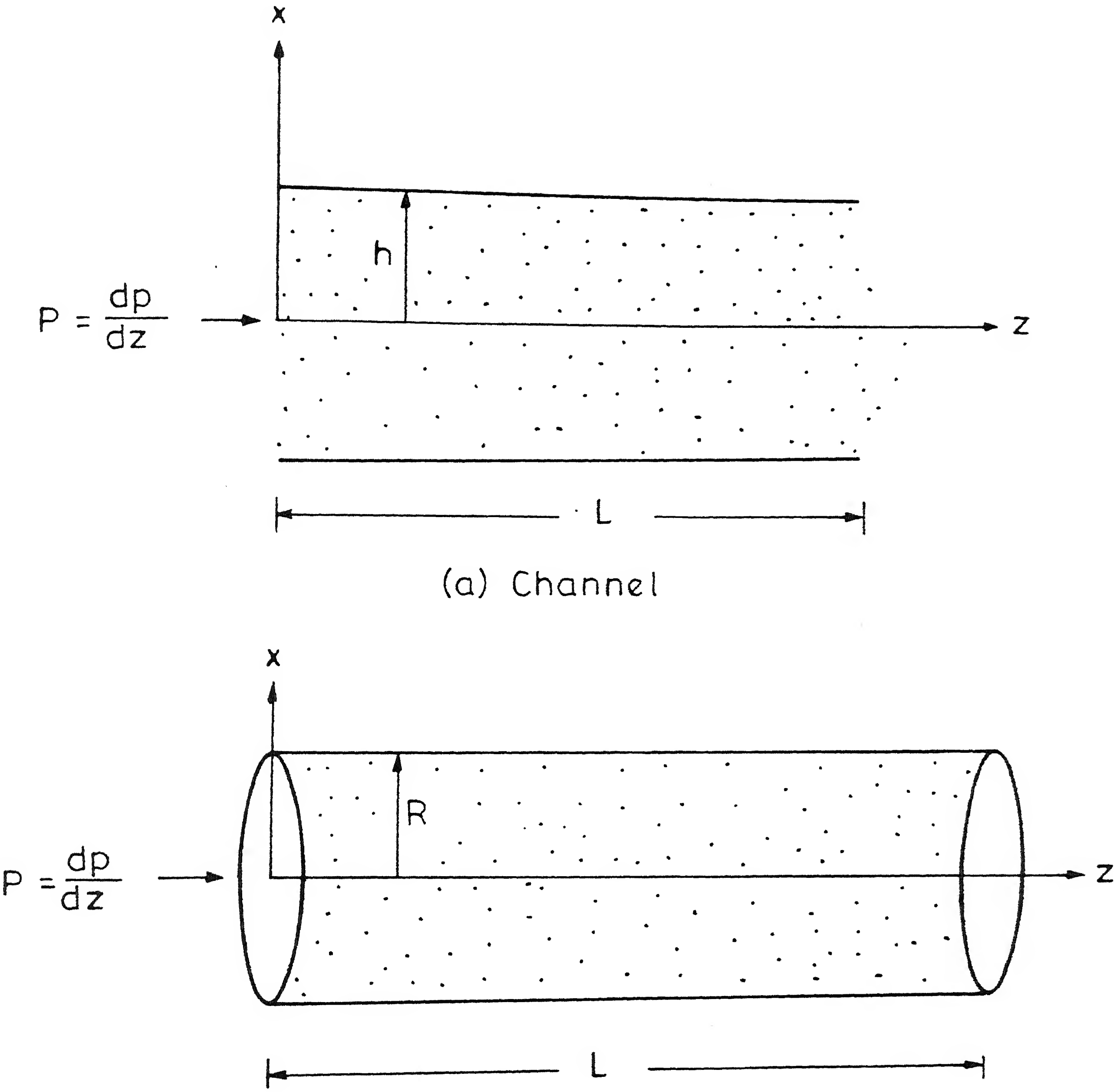
The effect of simple microfluid flow (in the core region) with a Newtonian fluid peripheral layer in a tube with very mild stenosis has been investigated. It is observed that resistance to the flow ($\bar{\lambda}$) and wall shear stress ($\bar{\tau}_w$) increase when $\bar{\delta}_s$ increases. These characteristics are further enhanced by certain combinations of simple microfluid parameters. Hence it may be inferred that, this suspension nature does not help blood flow in the functioning of the diseased arterial system.

CHAPTER - III

EFFECTS OF HETEROGENEOUS AND HOMOGENEOUS REACTIONS ON THE DISPERSION OF A SOLUTE IN SIMPLE MICROFLUID

3.1 INTRODUCTION

The dispersion of a solute in fluid flowing through channels/pipes is important in chemical as well as biological systems. In one of the early studies, Taylor (1953) presented a simple mathematical model to study dispersion of a solute through a fluid. He observed that, relative to a plane moving with the average speed of the flow, the solute disperses with an equivalent dispersion coefficient which depends upon (i) the average speed of the flow, (ii) the radius of the tube, and (iii) the molecular diffusion coefficient. In his analysis Taylor (1953) assumed that the solute does not chemically react with the fluid. However, subsequently many investigators analysed the dispersion problem by considering first order homogeneous reaction, under laminar flow conditions. Katz [1959] discussed the influence of the chemical reaction catalyzed on the wall of the tube. The combined effects of homogeneous and heterogeneous chemical reaction for a solute dispersing in Newtonian fluid flow have been discussed by Walker [1961], Solomon and Hudson [1967], Gupta and Gupta [1972] and others.



(a) Channel

(b) Tube

FIG.3.1 GEOMETRY OF THE PROBLEM

Here the equation is written by neglecting the axial diffusion and the term $-Kc$ represents volume rate of disappearance of the solute. D, is constant molecular diffusion coefficient, K is the first order homogeneous chemical reaction rate constant and v is the velocity of the simple microfluid along the z-axis.

Further, assuming that the walls of the channel are catalytic (hence, allow a heterogeneous chemical reaction) the differential material balance at the walls gives [Katz (1959), Gupta and Gupta (1972)],

$$\left.\begin{aligned} -D\frac{\partial c}{\partial x} &= fc \qquad \text{at} \quad x = h \\ -D\frac{\partial c}{\partial x} &= -fc \qquad \text{at} \quad x = -h \end{aligned}\right\} \tag{3.2}$$

where, fc gives the surface reaction rate.

For convenience the equation(3.1) is written relative to the axis moving with mean fluid velocity ($\bar{v}$) (with co-ordinates $z^{*} = z - \bar{v}t$, $x^{*} = x$) which on using the dimensionless parameters,

$$\theta = \frac{t\bar{v}}{L} \quad , \quad \bar{z} = \frac{z-\bar{v}t}{L} \quad , \quad \bar{x} = \frac{x}{h} \tag{3.3}$$

becomes

$$\frac{1}{t}\frac{\partial c}{\partial \theta} + \frac{v-\bar{v}}{L}\frac{\partial c}{\partial \bar{z}} = \frac{D}{h^{2}}\frac{\partial^{2} c}{\partial \bar{x}^{2}} - Kc \tag{3.4}$$

where L is the characteristic length along the flow direction and $\bar{v}$ is average fluid velocity given by,

$$\bar{v} = \frac{1}{h} \int_{o}^{h} v \, dx \, . \tag{3.5}$$

Assuming that the Taylor's limiting condition is valid i.e. the partial equilibrium is established in any cross-section of the channel, eqn. (3.4) reduces to

$$\frac{\partial^2 c}{\partial \bar{x}^2} - m^2 c = \frac{h^2}{DL} \frac{\partial c}{\partial \bar{z}} \, (v - \bar{v}) \tag{3.6}$$

where $\frac{\partial c}{\partial \bar{z}}$ is independent of $\bar{x}$, $\frac{\partial c}{\partial t} = 0$ and $m^2 = \frac{Kh^2}{D}$ (3.7)

characterizes the chemical reaction rate.

The boundary condition (3.2) can be rewritten as

$$\left.\begin{aligned} \frac{\partial c}{\partial \bar{x}} + \gamma c &= 0 \qquad \text{at} \quad \bar{x} = 1 \\ \frac{\partial c}{\partial \bar{x}} - \gamma c &= 0 \qquad \text{at} \quad \bar{x} = -1 \end{aligned}\right\} \tag{3.8}$$

where $\gamma = \frac{hf}{D}$ is the surface reaction rate constant.

Now to solve the equation (3.6) for concentration ,c, we need the expression of velocity ,v, of the simple microfluid. As, the flow is laminar and one dimensional we take $\vec{v} = (0,\ 0,\ v(z))$ and the non-zero components of the tensor ν_{kl} are $\nu_{13}(z)$ and $\nu_{31}(z)$ only, thus the governing equations [Chapter 1, (eqns. 1.9 to 1.12)] for the flow of simple microfluid in a channel can be written as [Chandra and Agarwal (1983)],

$$\frac{1}{h} v'' - E_1 \nu'_{(13)} + E_3 \left(\frac{v''}{2h} + \nu'_{(13)}\right) - hP = 0 \qquad (3.9)$$

$$(\bar{K}_1+\bar{K}_3)\,\nu''_{31} + (\bar{K}_2+\bar{K}_4)\,\nu''_{13} - 2(E_1+2E_2)\,\nu_{(13)} - E_3\left(\frac{v'}{h} + 2\,\nu_{[13]}\right) = 0 \qquad (3.10)$$

$$(\bar{K}_1-\bar{K}_3)\,\nu''_{31} + (\bar{K}_2-\bar{K}_4)\,\nu''_{13} - 2(E_1+2E_2)\,\nu_{(13)} + E_3\left(\frac{v'}{h} + 2\,\nu_{[13]}\right) = 0\,. \qquad (3.11)$$

Here $(\ )' = \frac{d}{d\bar{x}}$, $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$, $\bar{K}_4$ are defined as in Chapter I.

Here $\nu_{(13)}$, $\nu_{[13]}$ are the symmetric and skew-symmetric parts of ν_{13} respectively, as defined by equation (2.15).

The boundary conditions for v, ν_{13}, ν_{31} are taken as follows [Kang and Eingen (1976), Chandra and Agarwal (1983)]

$$\left.\begin{array}{llll} \text{(i)} & v = 0\ ; & \nu_{(13)} = \frac{A_1}{h} v'\ ; & \nu_{[13]} = \frac{A_2}{h} v' \quad \text{at } \bar{x}=1 \\ \text{(ii)} & v' = 0\ ; & \nu_{(13)} = 0\ ; & \nu_{[13]} = 0 \quad \text{at } \bar{x}=0 \end{array}\right\} \qquad (3.12)$$

Solving the above equations (3.9 to 3.11) as in Chapter II, the expressions for $\nu_{(13)}$, $\nu_{[13]}$ and v can be obtained as [Chandra and Agarwal (1983)],

$$\nu_{(13)} = -\frac{Ph}{2\mu}\left[\frac{b_1^*}{2}(1+f_1^*)\frac{\mathrm{Sinh}\bar{\alpha}\bar{x}}{\mathrm{Sinh}\bar{\alpha}} + \frac{b_2^*}{2}(1+f_2^*)\frac{\mathrm{Sinh}\bar{\beta}\bar{x}}{\mathrm{Sinh}\bar{\beta}}\right] \qquad (3.13)$$

$$\nu_{(13)} = -\frac{Ph}{2\mu}\left[\bar{x} + \frac{b_1^*}{2}(1+f_1^*)\frac{\mathrm{Sinh}\bar{\alpha}\bar{x}}{\mathrm{Sinh}\bar{\alpha}} + \frac{b_2^*}{2}(1-f_2^*)\frac{\mathrm{Sinh}\bar{\beta}\bar{x}}{\mathrm{Sinh}\bar{\beta}}\right] \qquad (3.14)$$

and

$$v = -\frac{Ph^2}{2\mu}\left[1-\bar{x}^2+d_1^*\,(\mathrm{Cosh}\bar{\alpha} - \cosh\bar{\alpha}\bar{x}) + d_2^*(\cosh\bar{\beta} - \cosh\bar{\beta}\bar{x})\right] \qquad (3.15)$$

where

$$\left.\begin{aligned}
b_1^* &= [-2A_1h_2^* + (1+2A_2)g_2^*]/[g_1^*h_2^* - g_2^*h_1^*] \\
b_2^* &= [-2A_1h_1^* + (1+2A_2)g_1^*]/[g_1^*h_2^* - g_2^*h_1^*] \\
d_i^* &= -\frac{b_i^*\;e_i^*}{a_i\sinh a_i} \\
g_i^* &= (-1)^{i+1}\left\{\frac{(1+f_i^*)}{2} - A_1e_i^*\right\} \\
h_i^* &= (-1)^{i+1}\left\{\frac{(1-f_i^*)}{2} - A_2e_i^*\right\} \\
f_i^* &= \frac{(a_i^2 - \bar{\alpha}_4^2)\bar{K}_1\bar{K}_4 + (a_i^2 - \bar{\alpha}_2^2)\,\bar{K}_2\bar{K}_3}{\bar{K}_1\bar{K}_3\;(\bar{\alpha}_1^2 - \bar{\alpha}_3^2)} \\
e_i^* &= \frac{(E_1 - E_3) + (E_1 + E_3)\,f_i^*}{(2 + E_3)}
\end{aligned}\right\} \qquad (3.16)$$

and $a_1 = \bar{\alpha}$; $a_2 = \bar{\beta}$ and other constants such as $\bar{\alpha}_i$, etc. are as given in Chapter II but non-dimensionalized by h instead of R_o.

Thus the average velocity of the fluid is obtained as

$$\bar{v} = -\frac{Ph^2}{2\mu}\left[\frac{2}{3} + d_1^*\left[\cosh\bar{\alpha} - \frac{\sinh\bar{\alpha}}{\bar{\alpha}}\right] + d_2^*\left[\cosh\bar{\beta} - \frac{\sinh\bar{\beta}}{\bar{\beta}}\right]\right] \quad (3.17)$$

Solving equation (3.6) with equations (3.15), (3.17) and the boundary conditions (3.12), we get

$$c = c^*\left\{\frac{2}{m^4} + \frac{\bar{x}^2}{m^2} - \frac{1}{m^2}\left[1 + \frac{6\cosh m\bar{x}}{F_4(\gamma,m)}\right.\right.$$

$$- \sum_{i=1}^{2} d_i^*\left\{\frac{\sinh a_i}{a_i}\left(\frac{1}{m^2} - \frac{a_i^2\cosh m\bar{x}}{m(a_i^2-m^2)\sinh m}\right) + \frac{\cosh a_i\bar{x}}{a_i^2 - m^2}\right\}\Bigg]\Bigg\}$$

$$+ \frac{c^*\gamma}{F_4(\gamma,m)}\left[\left\{2m\cosh m\bar{x}\, F_1(m,\bar{\alpha},\bar{\beta})\right\} - 2\sinh m\, F_2(m,\bar{\alpha},\bar{\beta})\right.$$

$$\left. + 2\gamma\sinh m\, F_1(m,\bar{\alpha},\bar{\beta})\right] \quad (3.18)$$

where

$$c^* = \frac{hv_o}{D}\frac{\partial c}{\partial z^*}, \quad v_o = \frac{Ph^2}{2\mu}$$

$$F_1(m,\bar{\alpha},\bar{\beta}) = \left[F_3(m,\bar{\alpha},\bar{\beta}) - \frac{1}{m^2} + \sum \frac{d_i^*}{a_i^2-m^2}\cosh a_i\right] \quad (3.19)$$

$$F_2(m,\bar{\alpha},\bar{\beta}) = \left[\frac{2}{m^2} - \sum \frac{d_i^* a_i}{a_i^2-m^2}\sinh a_i\right] \quad (3.20)$$

$$F_3(m,\bar{\alpha},\bar{\beta}) = \left[\frac{1}{m^2}\left(\frac{1}{3} - \frac{2}{m^2} + \sum \frac{d_i^*\sinh a_i}{a_i}\right)\right] \quad (3.21)$$

$$F_4(\gamma,m) = \left[\sinh m\ (m^2 + \gamma^2) + 2\gamma m \cosh m\right] \tag{3.22}$$

[Note : Throughout this chapter the summation index ,i, takes the value 1 and 2].

The average solute flux, Q, across the plane which moves with the mean speed of the flow can be obtained from,

$$Q = 2h \int_0^1 c\ (\bar{x})\ (v - \bar{v})\ d\bar{x}\ . \tag{3.23}$$

Substituting the expression of v and $\bar{v}$ from eqn. (3.15) and eqn. (3.17) in eqn. (3.23) we get,

$$\frac{Q}{2h} = \left(-\frac{v_o^2 h}{2D}\frac{\partial c}{\partial z^*}\right) M_1\ (m,\ \bar{\alpha},\ \bar{\beta},\ \gamma) \tag{3.24}$$

where

$$M_1\ (m,\bar{\alpha},\bar{\beta},\gamma) = \left[\frac{1}{3} + d_1^*\ \frac{\text{Sinh}\ \bar{\alpha}}{\bar{\alpha}} + d_2^*\ \frac{\text{Sinh}\ \bar{\beta}}{\bar{\beta}}\right] .$$

$$\left[F_5\ \frac{\text{Sinhm}}{m} - F_3(m,\bar{\alpha},\bar{\beta}) + \frac{1}{3m^3} - \frac{d_1^*\text{Sinh}\bar{\alpha}}{\bar{\alpha}(\bar{\alpha}^2-m^2)} - \frac{d_2^*\text{Sinh}\bar{\beta}}{\bar{\beta}(\bar{\beta}^2-m^2)}\right]$$

$$-\left[F_5\ \left(\frac{1}{m}\ \text{Sinhm} - \frac{2}{m^2}\ \text{Coshm} + \frac{2}{m^3}\ \text{Sinhm}\right)\right.$$

$$\left. - F_3\ (m,\bar{\alpha},\bar{\beta}) + \frac{1}{3m^2} - \frac{d_1^*\text{Sinh}\bar{\alpha}}{\bar{\alpha}(\bar{\alpha}^2-m^2)} - \frac{d_2^*\ \text{Sinh}\bar{\beta}}{\bar{\beta}(\bar{\beta}^2-m^2)}\right]$$

$$+ \sum_{i=1}^{2} \left[F_3(m,\bar{\alpha},\bar{\beta})\left(\frac{d_i^* \ \mathrm{Sinh}a_i}{a_i}\right) - \frac{d_i^* F_5}{2} \left\{ \frac{\mathrm{Sinh}(a_i+m)}{a_i+m} + \frac{\mathrm{Sinh}(a_i-m)}{(a_i-m)} \right\} \right.$$

$$- \frac{1}{m^2} \left[\frac{1}{a_i} \mathrm{Sinh}a_i - \frac{2}{a_i^2} \mathrm{Cosh}a_i + \frac{2}{a_i^3} \mathrm{Sinh}a_i \right]$$

$$+ \frac{d_1^* d_2^*}{\bar{\alpha}} \left[\frac{\mathrm{Sinh}(\bar{\alpha}+\bar{\beta})}{\bar{\alpha}+\bar{\beta}} + \frac{\mathrm{Sinh}(\bar{\alpha}-\bar{\beta})}{(\bar{\alpha}-\bar{\beta})} \left(\frac{1}{a_i^2 - m^2}\right) \right]$$

$$\left. + \frac{d_i^{*2}}{(a_i^2 - m^2)} \left[\frac{1}{4a_i} - \mathrm{Sinh}\ 2a_i + \frac{1}{2} \right] \right] \tag{3.25}$$

and

$$F_5 = \frac{1}{F_4(\gamma,m)} \left[\left\{ 2\gamma m F_1(m,\bar{\alpha},\bar{\beta}) - 2mF_2(m,\bar{\alpha},\bar{\beta}) \right\} \mathrm{Cosh}m \right.$$

$$\left. - \left\{ 2\gamma F_2(m,\bar{\alpha},\bar{\beta}) - 2\gamma^2 F_1(m,\bar{\alpha},\bar{\beta}) \right\} \mathrm{Sinh}m \right] \tag{3.26}$$

Now comparing the equation (3.24) with the Fick's law of diffusion, namely

$$J^* = - D^* \frac{\partial c}{\partial z} \tag{3.27}$$

we get the equivalent dispersion coefficient D^* with which the solute disperses relative to a plane moving with the mean speed of the flow is obtained as follows

$$D^* = \frac{v_o^2 h^2}{2D} M_1(m,\bar{\alpha},\bar{\beta},\gamma) \tag{3.28}$$

3.3 PART II : FLOW THROUGH PIPES

In this section we consider the combined effects of homogeneous and heterogeneous chemical reaction on the dispersion of a solute in a simple microfluid flowing through a circular hollow cylindrical tube of radius R (fig. 3.1b). We shall analyse this problem under the same assumptions as discussed in Part I.

Thus, the non-dimensional form of concentration equation in moving coordinates under Taylor's limiting condition can be written as,

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial c}{\partial \bar{r}}\right) - m^2 c = \left(\frac{R}{D}\frac{\partial c}{\partial z^*}\right)(v-\bar{v}) \qquad (3.29)$$

where, $m^2 = \frac{KR^2}{D}$, $z^* = (z-\bar{v}t)/R$, $\frac{\partial C}{\partial z^*}$ is independent of r and $\frac{\partial c}{\partial t}=0$.

The boundary conditions are [Katz (1959)],

$$\left.\begin{aligned} \frac{\partial c}{\partial \bar{r}} &= 0 \qquad \text{at} \quad \bar{r} = 0 \\ \frac{\partial c}{\partial \bar{r}} + \gamma c &= 0 \qquad \text{at} \quad \bar{r} = 1 \end{aligned}\right\} \qquad (3.30)$$

where

$$\gamma = \frac{Rf}{D} .$$

For this one dimensional flow under the uniform pressure gradient $\frac{dp}{dz} = P$, the velocity, v of a simple microfluid is given by eqn. (2.42) [in Chapter II, with $\delta=0$ and $R_1(z) = R(z) = R_o$].

Thus

$$v = \frac{PR_o^2}{4\mu}\left[(\bar{r}^2-1) - \left\{d_1\left\{I_o(\bar{\alpha})-I_o(\bar{\alpha}\bar{r})\right\} + d_2\left\{I_o(\bar{\beta})-I_o(\bar{\beta}\bar{r})\right\}\right\}\right] \quad (3.31)$$

The expressions for d_1, d_2 and other constants are as given in eqn. (2.43). The average velocity of the flow, $\bar{v}$, is given by,

$$\bar{v} = \frac{1}{\pi R^2}\int_o^R 2\pi r v \, dr$$

$$= v_o\left[\frac{1}{2} + d_1\left\{I_o(\bar{\alpha}) - \frac{2I_1(\bar{\alpha})}{\bar{\alpha}}\right\} + d_2\left\{I_o(\bar{\beta}) - \frac{2I_1(\bar{\beta})}{\bar{\beta}}\right\}\right] \quad (3.32)$$

where

$$v_o = PR_o^2/4\mu .$$

From eqns. (3.31) and (3.32) we get,

$$v-\bar{v} = v_o\left[\frac{1}{2} - \bar{r}^2+d_1\left\{\frac{2I_1(\bar{\alpha})}{\bar{\alpha}} - I_o(\bar{\alpha}\bar{r})\right\} + d_2\left\{\frac{2I_1(\bar{\beta})}{\bar{\beta}} - I_o(\bar{\beta}\bar{r})\right\}\right]$$

$$= v_o\, G_1(\bar{r}) \quad \text{(say).} \quad (3.33)$$

Using eqn. (3.33) the equation governing the solute concentration can be written as,

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{r}\frac{\partial c}{\partial \bar{r}}\right]-m^2c = \frac{Rv_o}{D}\frac{\partial c}{\partial z^*}\left[\frac{1}{2} - \bar{r}^2+ d_1\left\{\frac{2I_1(\bar{\alpha})}{\bar{\alpha}} - I_o(\bar{\alpha}\bar{r})\right\}\right.$$

$$\left. + d_2\left\{\frac{2I_1(\bar{\beta})}{\bar{\beta}} - I_o(\bar{\beta}\bar{r})\right\}\right] \quad (3.34)$$

Thus the solution of the above eqn. (3.34) alongwith the boundary conditions (3.30) gives the concentration of the solute in the following form :

$$c(\bar{r}) = c^{**}\left\{\frac{4}{m^4}\left[1 - \frac{\gamma I_o(m\bar{r})}{F_6(\gamma,m)}\right] + \frac{\bar{r}^2}{m^2} - \frac{1}{2m^2}\left[1 - \frac{\gamma I_o(m\bar{r})}{F_6(\gamma,m)}\right] - \frac{I_o(m\bar{r})}{F_6(\gamma,m)}\frac{(2+\gamma)}{m^2}\right.$$

$$- \sum_{i=1}^{2} d_i\left[\frac{2I_1(a_i)}{a_i m^2} - \left(\frac{I_o(m\bar{r})}{F_6(\gamma,m)}\right)\left\{\frac{a_i I_1(a_i) + \gamma I_o(a_i)}{a_i^2 - m^2}\right.\right.$$

$$\left.\left.\left. + \frac{2\gamma}{m^2}\frac{I_1(a_i)}{a_i}\right\} + \frac{I_o(a_i\bar{r})}{a_i^2 - m^2}\right]\right\} \qquad (3.35)$$

where

$$c^{**} = \frac{Rv_o}{D}\frac{\partial c}{\partial z^*} \quad \text{and} \quad F_6(\gamma,m) = mI_1(m) + \gamma I_o(m)$$

The average solute flux $\bar{Q}$ across the plane which moves with the mean speed of the flow can be obtained from,

$$\bar{Q} = \int_o^1 \bar{r}\,(v - \bar{v})\,c(\bar{r})\,d\bar{r} \qquad (3.36)$$

by substituting the expression for v, $\bar{v}$ from equations (3.31) and (3.32) respectively. Thus we get,

$$\bar{Q} = -\left(\frac{Rv_o^2}{D}\frac{\partial c}{\partial z^*}\right) M_2\,(m,\ \gamma,\ \bar{\alpha},\ \bar{\beta})$$

where

$$M_2(m,\gamma,\bar{\alpha},\bar{\beta}) = \bar{H}_o(m,\gamma) + d_1\ \bar{H}_1(m,\gamma,\bar{\alpha}) + d_2\ \bar{H}_1(m,\gamma,\bar{\beta})$$

$$+ d_1 d_2 \left[\bar{H}_2(m,\gamma,\bar{\alpha},\bar{\beta}) + \bar{H}_2(m,\gamma,\bar{\beta},\bar{\alpha})\right]$$

$$+ d_1^2 \left[\bar{H}_3(m,\gamma,\bar{\alpha})\right] + d_2^2 \left[\bar{H}_3(m,\gamma,\bar{\beta})\right] \qquad (3.37)$$

where

$$\bar{H}_o(m,\gamma) = \left\{\left[\frac{4\gamma}{m^4} + \frac{2+\gamma}{m^2} - \frac{\gamma}{2m^2}\right]\left[\frac{1}{F_6(\gamma,m)}\right]\right\}\left[\frac{I_1(m)}{m}\right]\left[\frac{1}{2} + \frac{4}{m^2}\right]$$

$$- \frac{1}{m^2}\left[\frac{1}{24} - 2I_o(m)\right] \qquad (3.38)$$

$$\bar{H}_1(m,\gamma,\bar{\alpha}) = \frac{1}{m^2\ \bar{\alpha}^2}\left\{2I_o(\bar{\alpha}) - \frac{4I_1(\bar{\alpha})}{\bar{\alpha}}\right\} - \frac{I_1(\bar{\alpha})}{2m^2\bar{\alpha}} +$$

$$\left\{\frac{mI_1(m)I_o(\bar{\alpha}) - \bar{\alpha}I_1(\bar{\alpha})\ I_o(m)}{m^2 - \bar{\alpha}^2} - \frac{2I_1(\bar{\alpha})I_1(m)}{m\bar{\alpha}}\right\}\frac{(8\gamma + 4m^2 + \gamma^2 m^2)}{2m^4\ F_6(\gamma,m)}$$

$$- \frac{1}{(\bar{\alpha}^2 - m^2)}\left[\frac{2I_o(\bar{\alpha})}{\bar{\alpha}^2} - \frac{I_1(\bar{\alpha})}{2\bar{\alpha}} - \frac{4I_1(\bar{\alpha})}{\bar{\alpha}^3}\right]\left[\frac{\bar{\alpha}I_1(\bar{\alpha}) + \gamma\ I_o(\bar{\alpha})}{\bar{\alpha}^2 - m^2}\right.$$

$$\left. + \frac{2\gamma\ I_1(\bar{\alpha})}{m^2\bar{\alpha}}\right]\left[\frac{2I_o(m)}{m^2} - \frac{I_1(m)}{2m} - \frac{4I_1(m)}{m^3}\right]\left[\frac{1}{F_6(\gamma,m)}\right] \qquad (3.39)$$

3.4 RESULTS AND DISCUSSION

It may be pointed out here, that for the particular case when γ =0 the equations (3.18) and (3.35) reduce to the case of Chandra and Agarwal (1983), while m=0 refers to the situation when the dispersion does not undergo any irreversible chemical reaction in the fluid, but has surface reaction at the wall.

The effect of various parameters such as chemical reaction rate constant, m, surface reaction rate constant, γ, and the viscosity coefficients E_1, E_2, E_3 on the equivalent dispersion coefficient can be seen through the function M_1, in the case of channel flow and M_2 for the pipe flow. Hence these expressions have been numerically calculated using Simpson's rule for different sets of values of various parameters. The results are presented by choosing the coefficients K_i's as $\bar{K}_1$ = 1.1, $\bar{K}_2$ = 1.0, $\bar{K}_3$ = -0.9 and $\bar{K}_4$ = 0.6 and A_1 = A_2 = 0.25.

The effect of various parameters on M_1 are shown in Figs. (3.2 to 3.9) and on M_2 in Figs. (3.10 to 3.14). Fig. (3.2) shows that, for a given simple microfluid, M_1 decreases as the chemical reaction rate constant 'm' increases. This effect is enhanced as the surface reaction rate constant increases. The effect of γ on M_1 is further elaborated in Figs. (3.6 to 3.9). It is observed from these figures that M_1 shows slight decrease as γ increases from 0 to 2 and then approaches asymptotic value as γ is increased beyond 2. This increase in M_1 with γ, becomes more appreciable as m increases. In the case of the tube similar

$$\bar{H}_2(m,\gamma,\bar{\alpha},\bar{\beta}) = \frac{2I_1(\bar{\alpha})}{\bar{\alpha}} \left[\frac{\bar{\beta}I_1(\bar{\beta}) + \gamma I_o(\bar{\beta})}{(\bar{\beta}^2 - m^2)} + \frac{2\gamma I_1(\bar{\beta})}{m^2\bar{\beta}}\right]\left[\frac{I_1(m)}{F_6(\gamma,m)}\right]$$

$$- \frac{2I_1(\bar{\alpha})\ I_1(\bar{\beta})}{\bar{\alpha}\bar{\beta}\ (\bar{\beta}^2 - m^2)} + \left[\frac{\bar{\alpha}I_1(\bar{\alpha})\ I_o(\bar{\beta}) - \bar{\beta}I_1(\bar{\beta})\ I_o(\bar{\alpha})}{(\bar{\alpha}^2 - \bar{\beta}^2)\ (\bar{\beta}^2 - \bar{\alpha}^2)}\right] \tag{3.40}$$

$$-\left[\frac{\bar{\beta}I_1(\bar{\beta}) + \gamma I_o(\bar{\beta})}{(\bar{\beta}^2 - m^2)} + \frac{2\gamma I_1(\bar{\beta})}{m^2\bar{\beta}}\right]\left[\frac{\bar{\alpha}I_1(\bar{\alpha})I_o(m) - mI_1(m)I_o(\bar{\alpha})}{(\bar{\alpha}^2 - m^2)\ F_6(\gamma,m)}\right]$$

$$\bar{H}_3(m,\gamma,\bar{\alpha}) = \frac{2I_1(\bar{\alpha})}{\bar{\alpha}} \left[\frac{\bar{\alpha}I_1(\bar{\alpha}) + \gamma I_o(\bar{\alpha})}{(\bar{\alpha}^2 - m^2)} + \frac{2\gamma I_1(\bar{\alpha})}{m^2\bar{\alpha}}\right]\left[\frac{I_1(m)}{F_6(\gamma,m)}\right]$$

$$-\left[\frac{\bar{\alpha}I_1(\bar{\alpha}) + \gamma I_o(\bar{\alpha})}{(\bar{\alpha}^2 - m^2)} + \frac{2\gamma I_1(\bar{\alpha})}{m^2\bar{\alpha}}\right]\left[\frac{\bar{\alpha}I_1(\bar{\alpha})I_o(m) - mI_1(m)I_o(\bar{\alpha})}{(\bar{\alpha}^2 - m^2)\ F_6(\gamma,m)}\right]$$

$$- \frac{2I_1^2(\bar{\alpha})}{\bar{\alpha}^2(\bar{\alpha}^2-m^2)} + \frac{1}{(\bar{\alpha}^2-m^2)}\left[\frac{1}{2}\ I_o^2(\bar{\alpha}) - I_1^2(\bar{\alpha})\right] \tag{3.41}$$

As usual comparing eqn. (3.37) with Fick's law of diffusion we find the equivalent dispersion coefficient D^* with which the solute disperses relative to a plane moving with the mean velocity of flow, D^* is given by

$$D^* = \left[\frac{v_o^2\ R^2}{D}\right] M_2\ (m,\gamma,\bar{\alpha},\bar{\beta}) \tag{3.42}$$

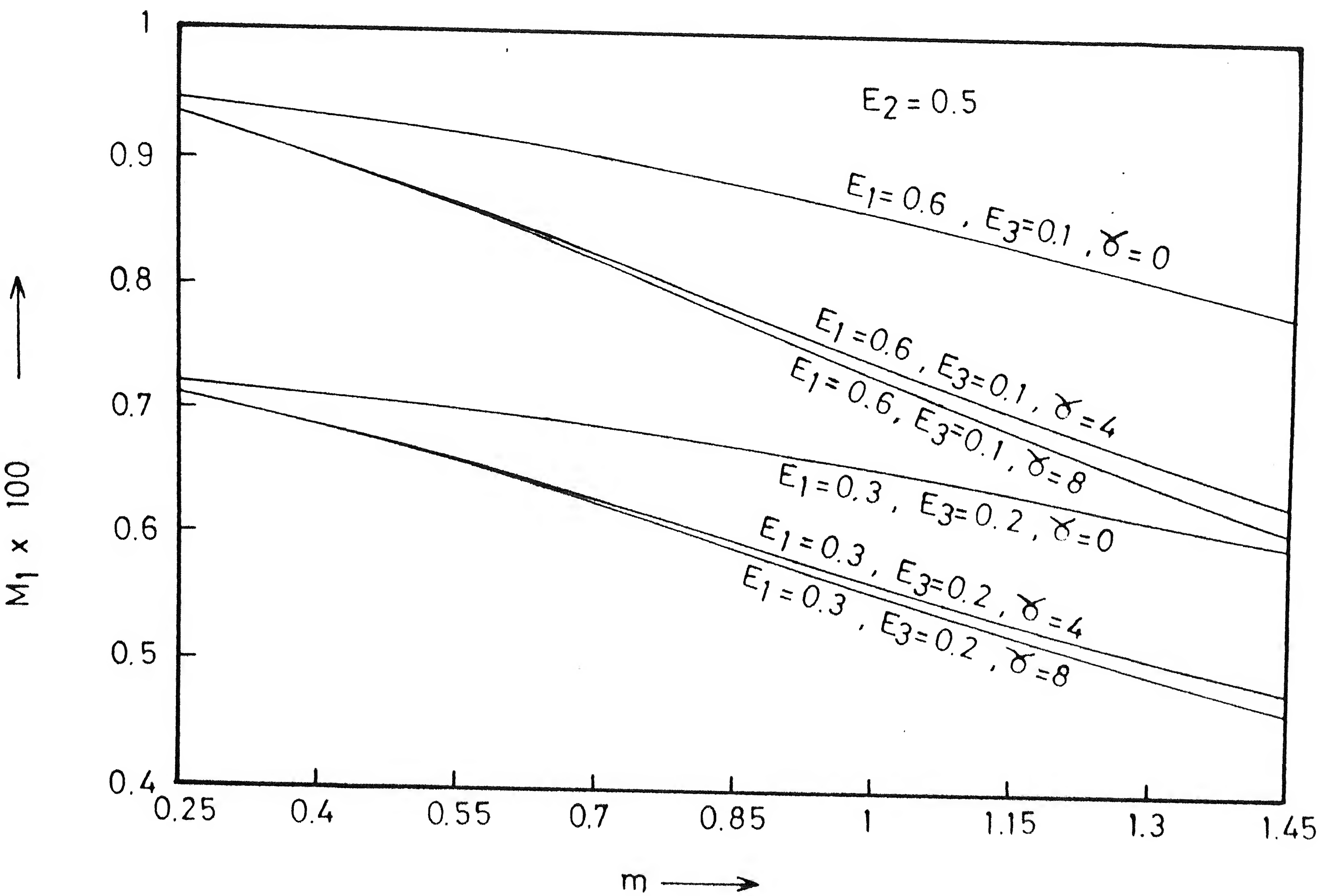


FIG. 3.2 VARIATION OF M_1 WITH m FOR DIFFERENT γ , E_1 , E_3 .

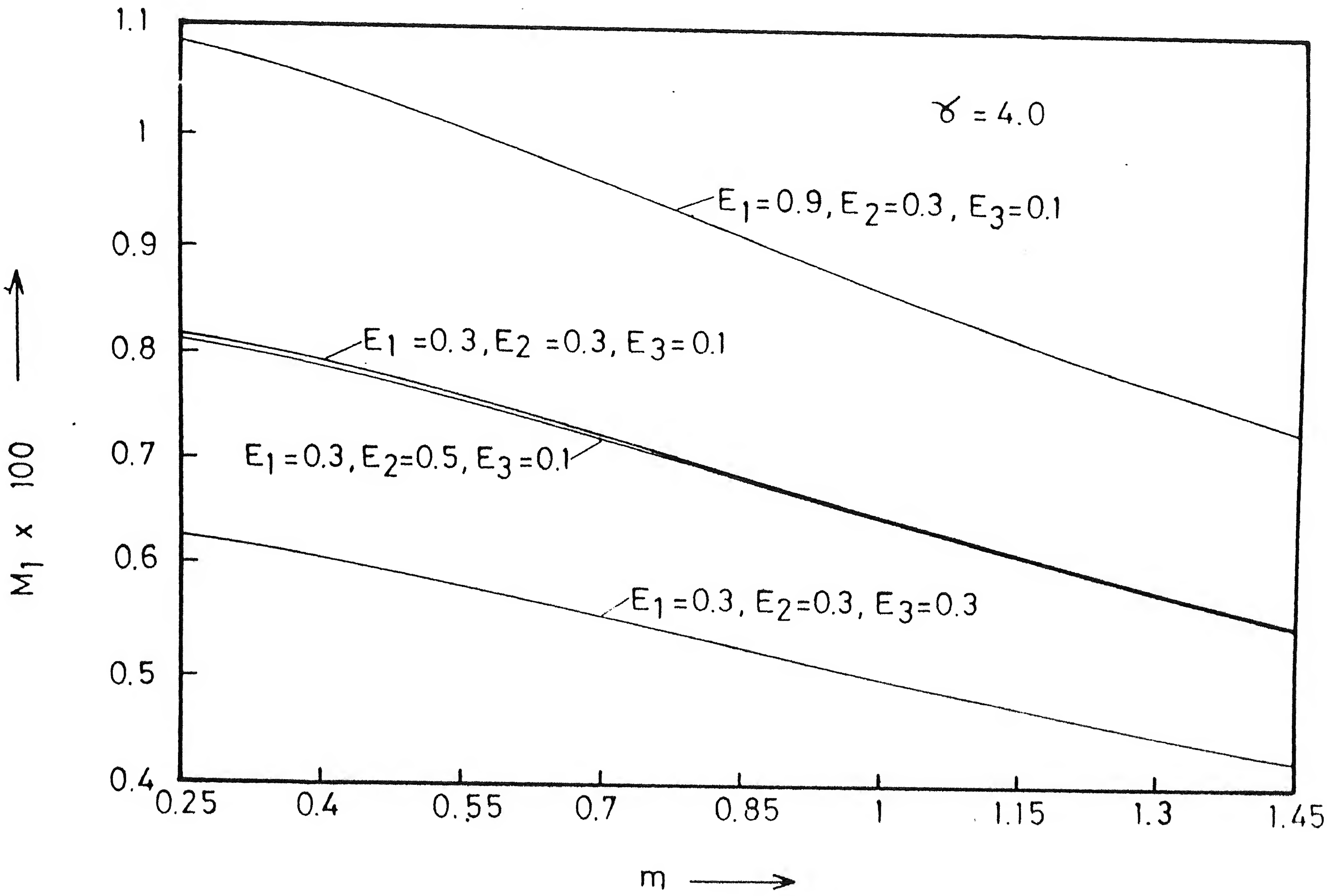


FIG. 3.3 VARIATION OF M_1 WITH m FOR DIFFERENT E_1, E_2 & E_3.

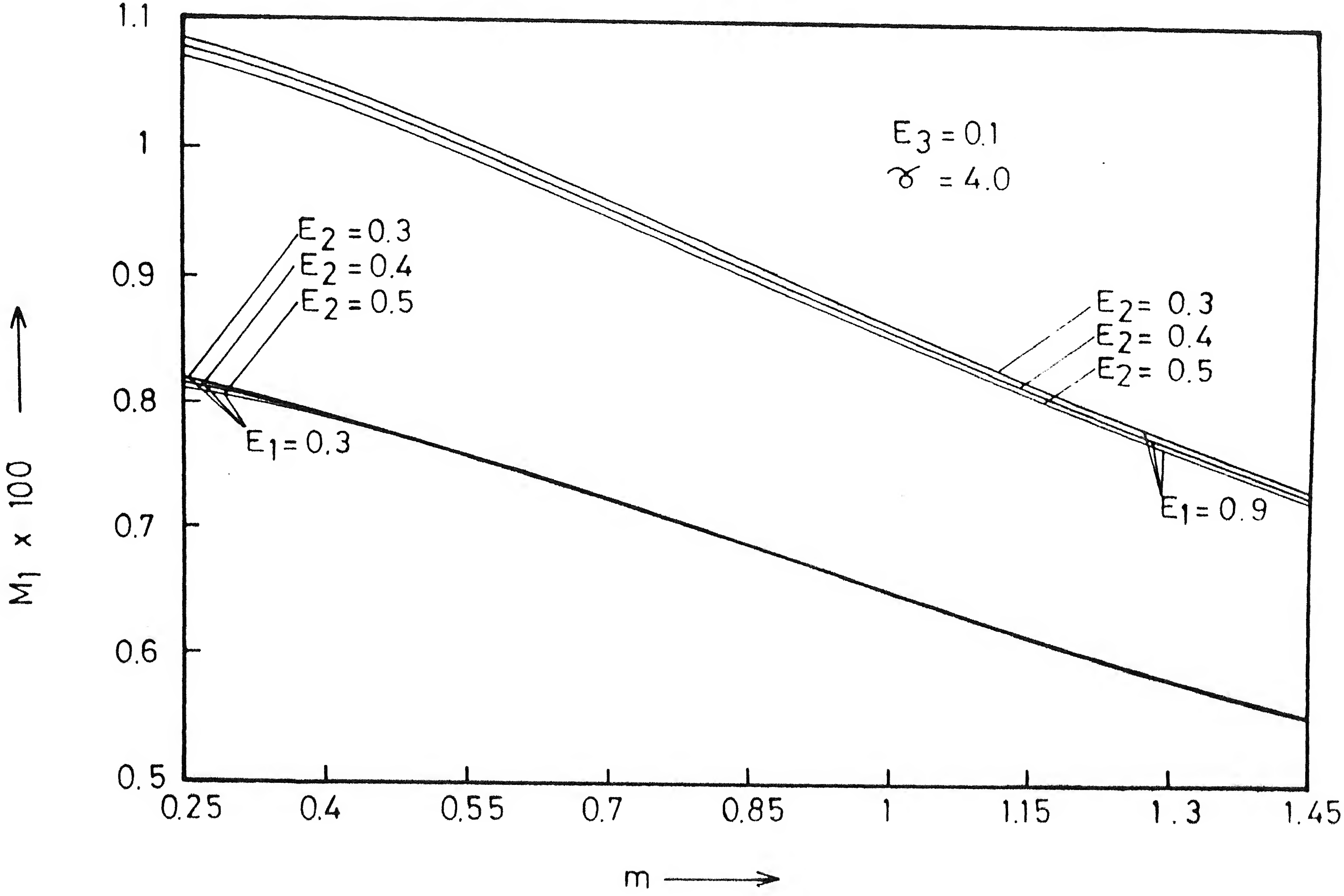


FIG.3.4 VARIATION OF M_1 WITH m FOR DIFFERENT E_1 & E_2.

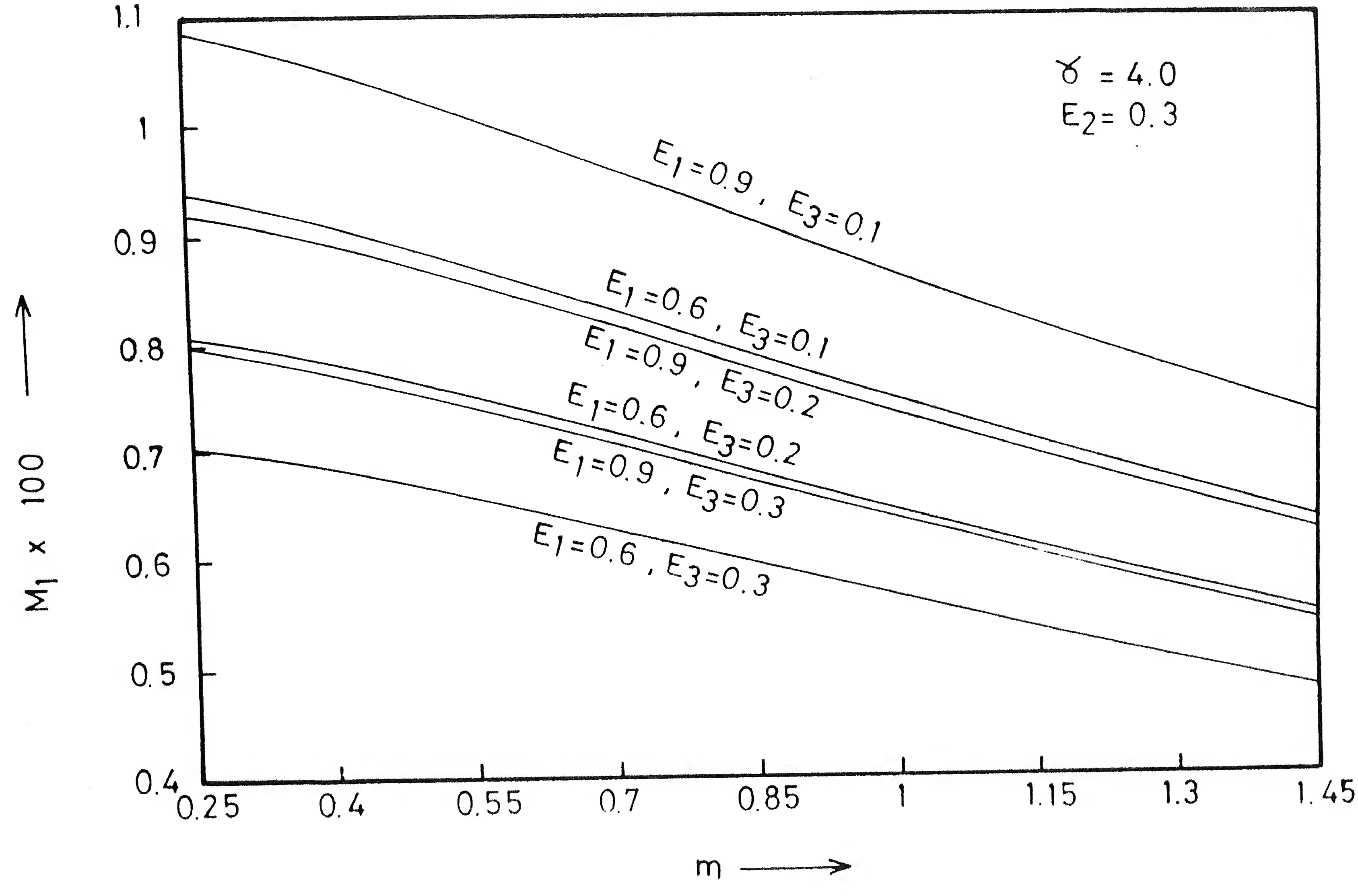


FIG.3.5 VARIATION OF M_1 WITH m FOR DIFFERENT E_1 & E_3.

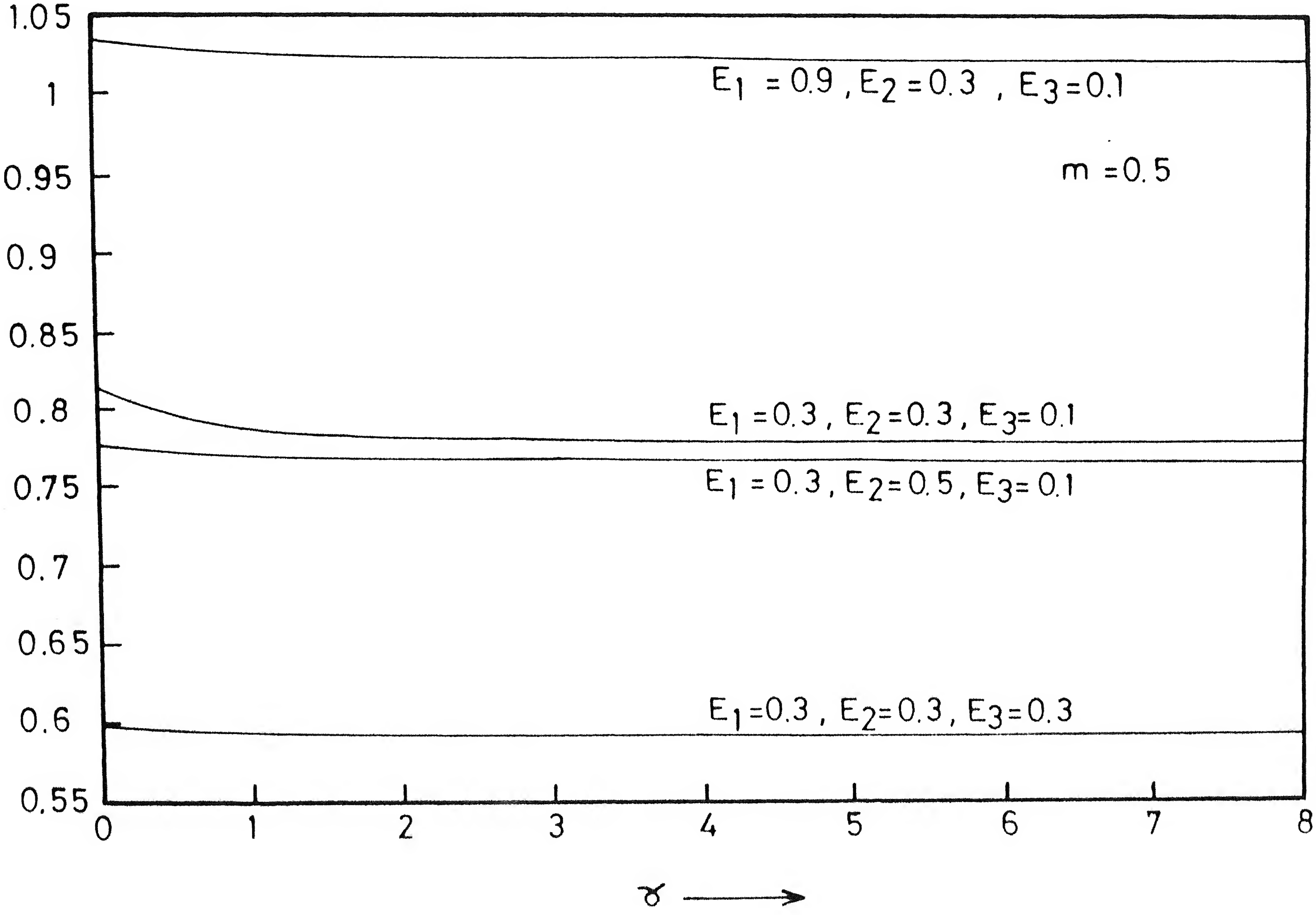


FIG. 3.6 VARIATION OF M_1 WITH γ FOR DIFFERENT E_1, E_2 & E_3.

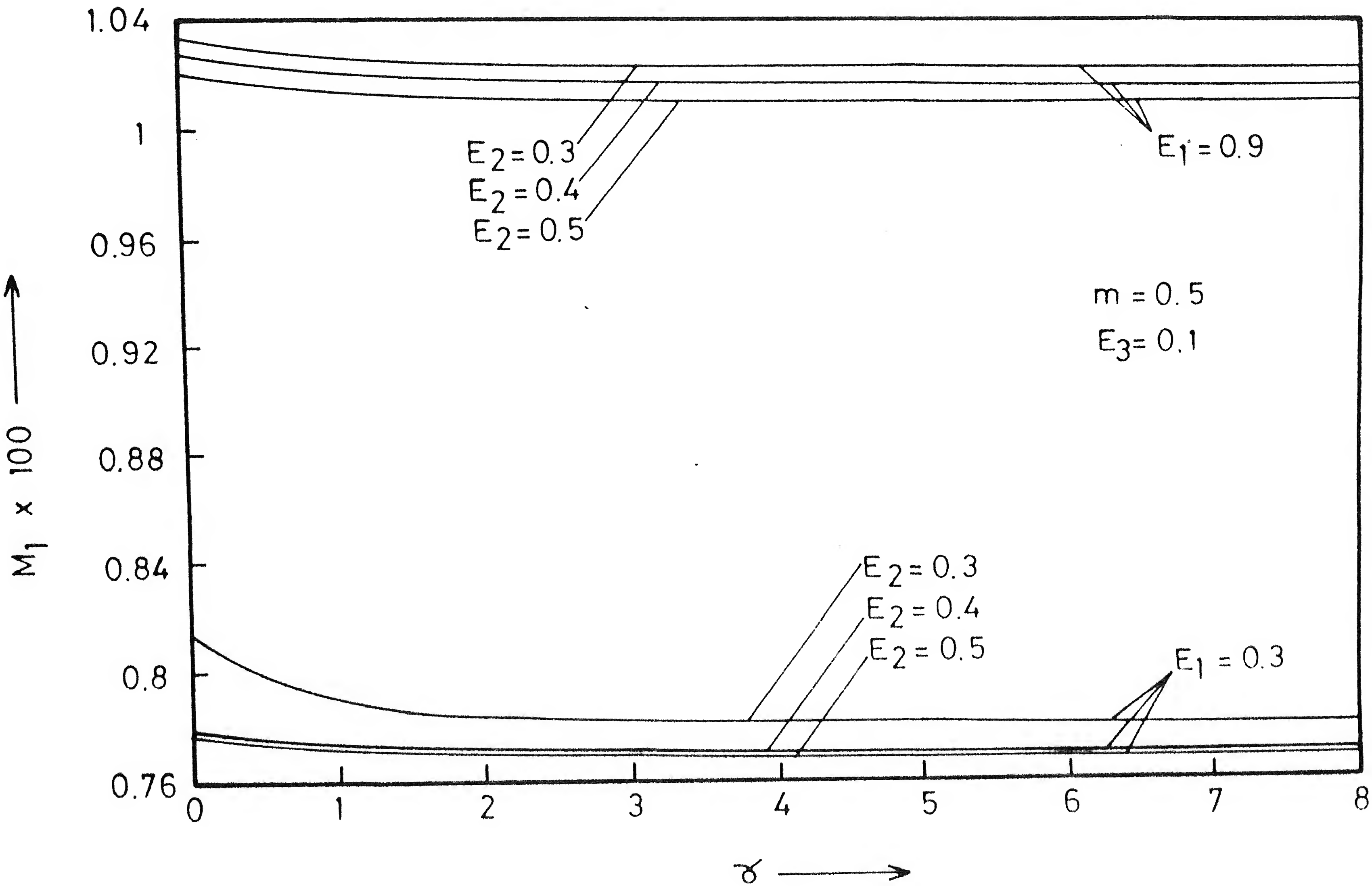


FIG.3.7 VARIATION OF M_1 WITH γ FOR DIFFERENT E_1 & E_2 .

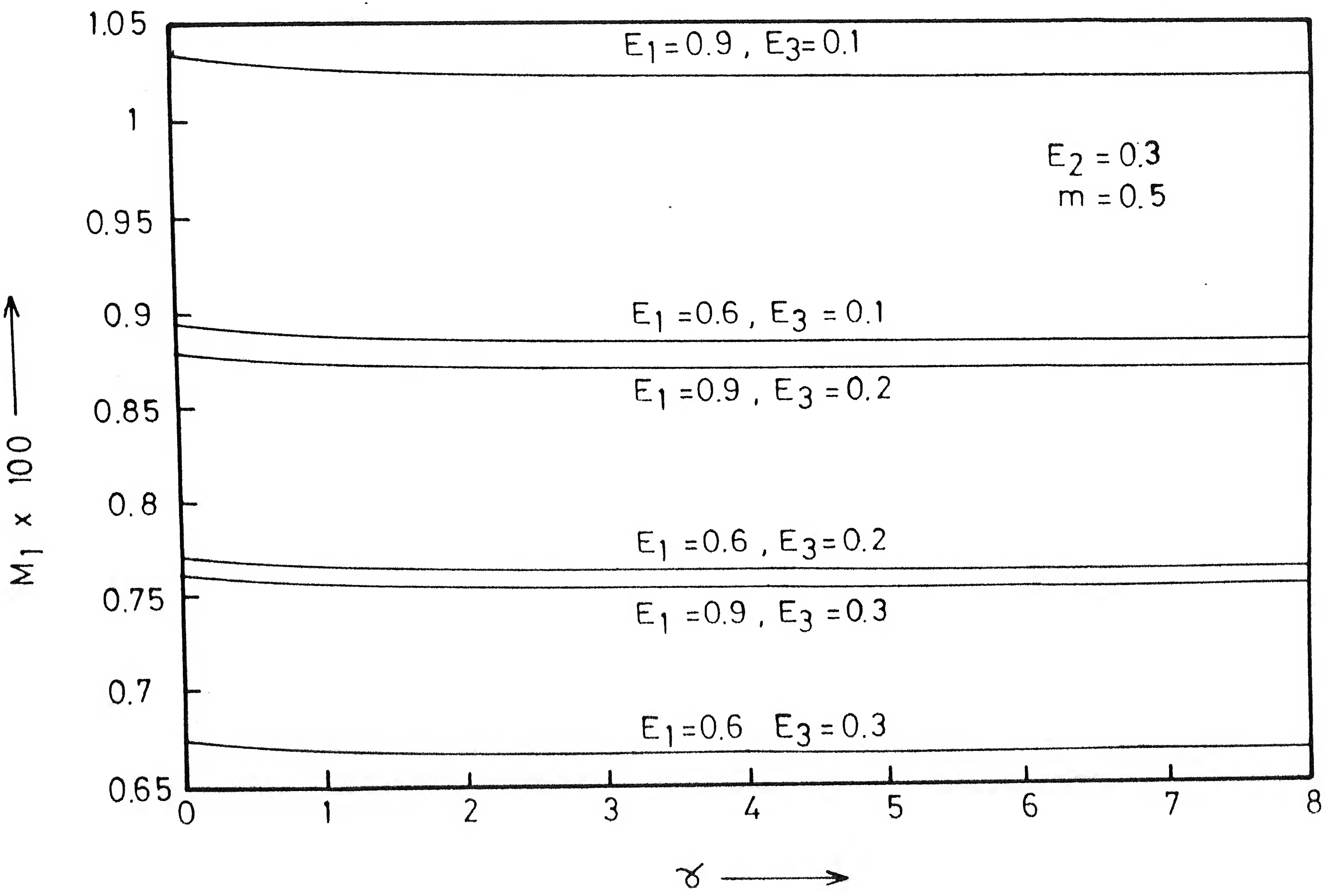


FIG.3.8 VARIATION OF M_1 WITH γ FOR DIFFERENT E_1 & E_3.

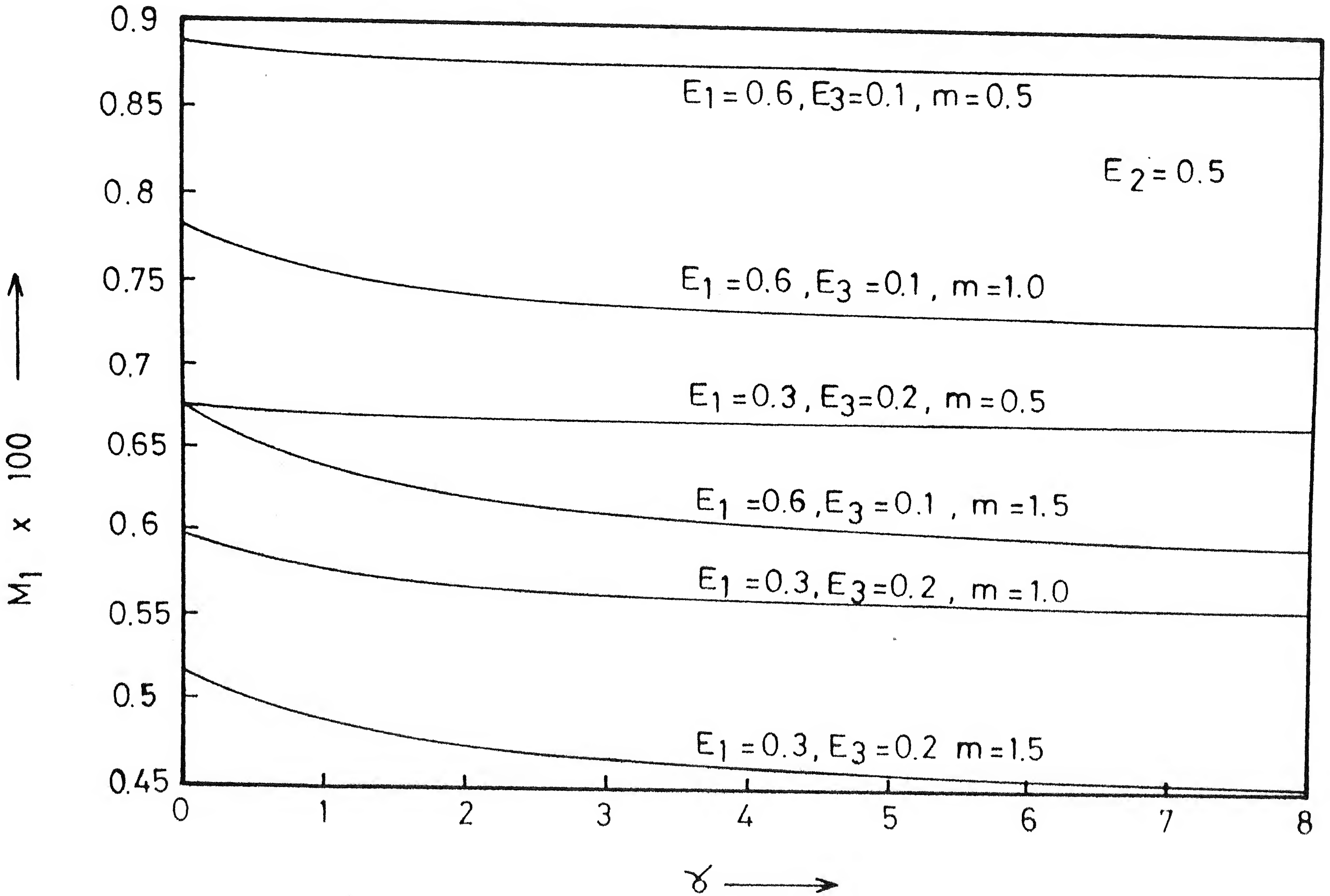


FIG. 3.9 VARIATION OF M_1 WITH γ FOR DIFFERENT m , E_1 & E_3 .

effects of surface reaction rate constant γ on M_2 are observed (Figs. 3.10 to 3.13).

The effects of simple microfluid parameters in the presence of surface reaction at the wall are depicted in Figs. (3.3 to 3.5) (with γ = 4.0). These figures show that M_1 increases as the viscosity coefficient E_1 increases, but decreases as the viscosity coefficient E_2 and E_3 increase. However, variation of M_1 with E_2 is not very significant and is almost negligible for smaller values of E_1. This behaviour is similar to the case of no surface reaction rate at the wall (γ = 0) [Chandra and Agarwal (1983)]. Combined effects of the viscosity coefficients E_1, E_2, E_3 and γ for m = 0.5 can also be observed through Figs. (3.7 to 3.9). In case of tube flow the variation of M_2 with respect to various parameter shows similar behaviour as has been observed in the case of channel flow. However, M_2 approaches asymptotic values for higher values of γ (around γ = 5). Also unlike the channel case here, the effect of E_2 on M_2 is not very insignificant.

3.5 CONCLUSION

The combined effect of homogeneous and heterogeneous reaction rate constants on D^* is to decrease it for a given set of simple microfluid parameters. The decrease in D^* with homogeneous reaction rate becomes more in the presence of surface reaction on the walls. The equivalent dispersion coefficient decreases as the simple microfluid parameters E_2 and E_3 increases but it increases with the parameter E_1.

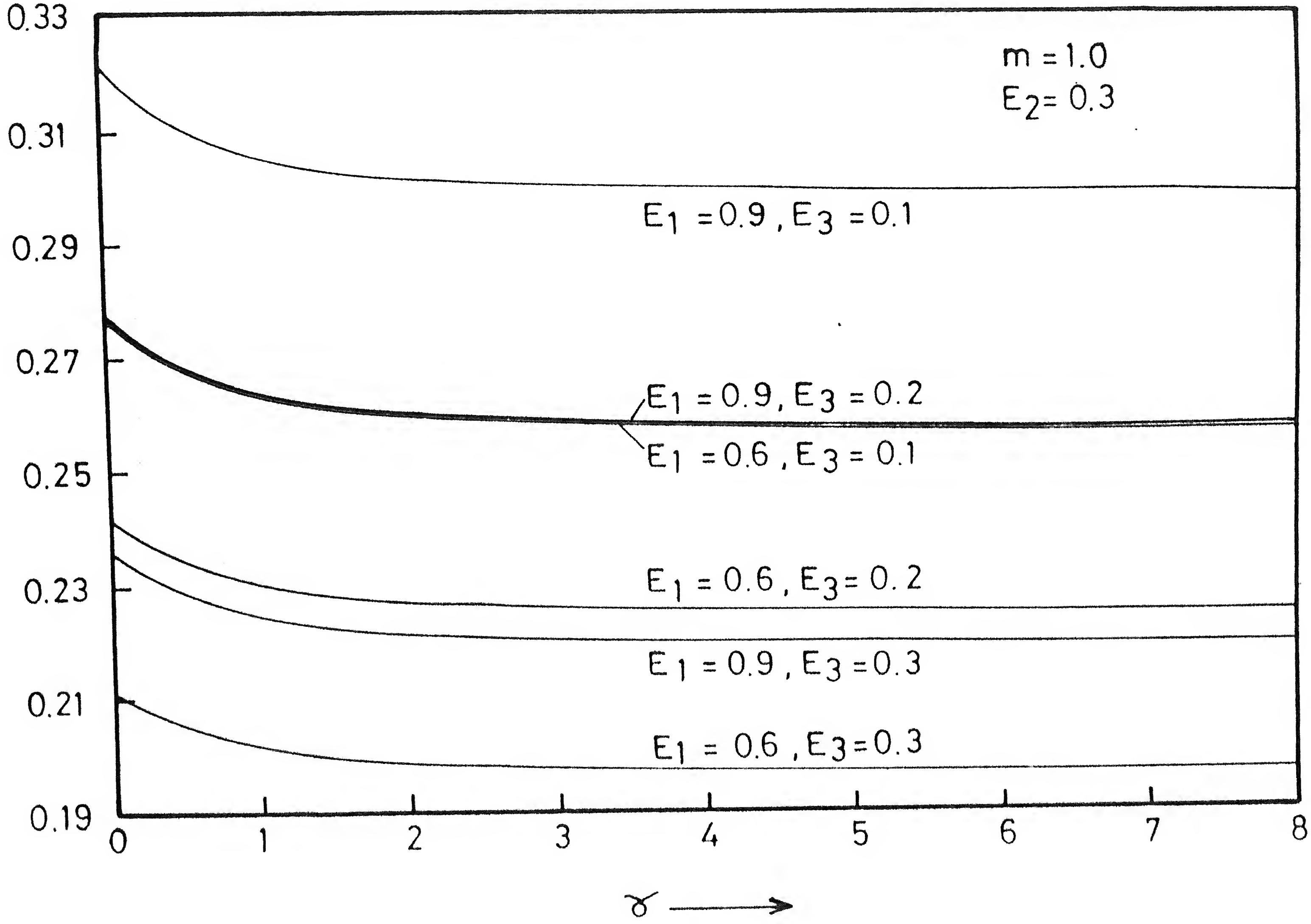


FIG.3.10 VARIATION OF M_2 WITH γ FOR DIFFERENT E_1 & E_3.

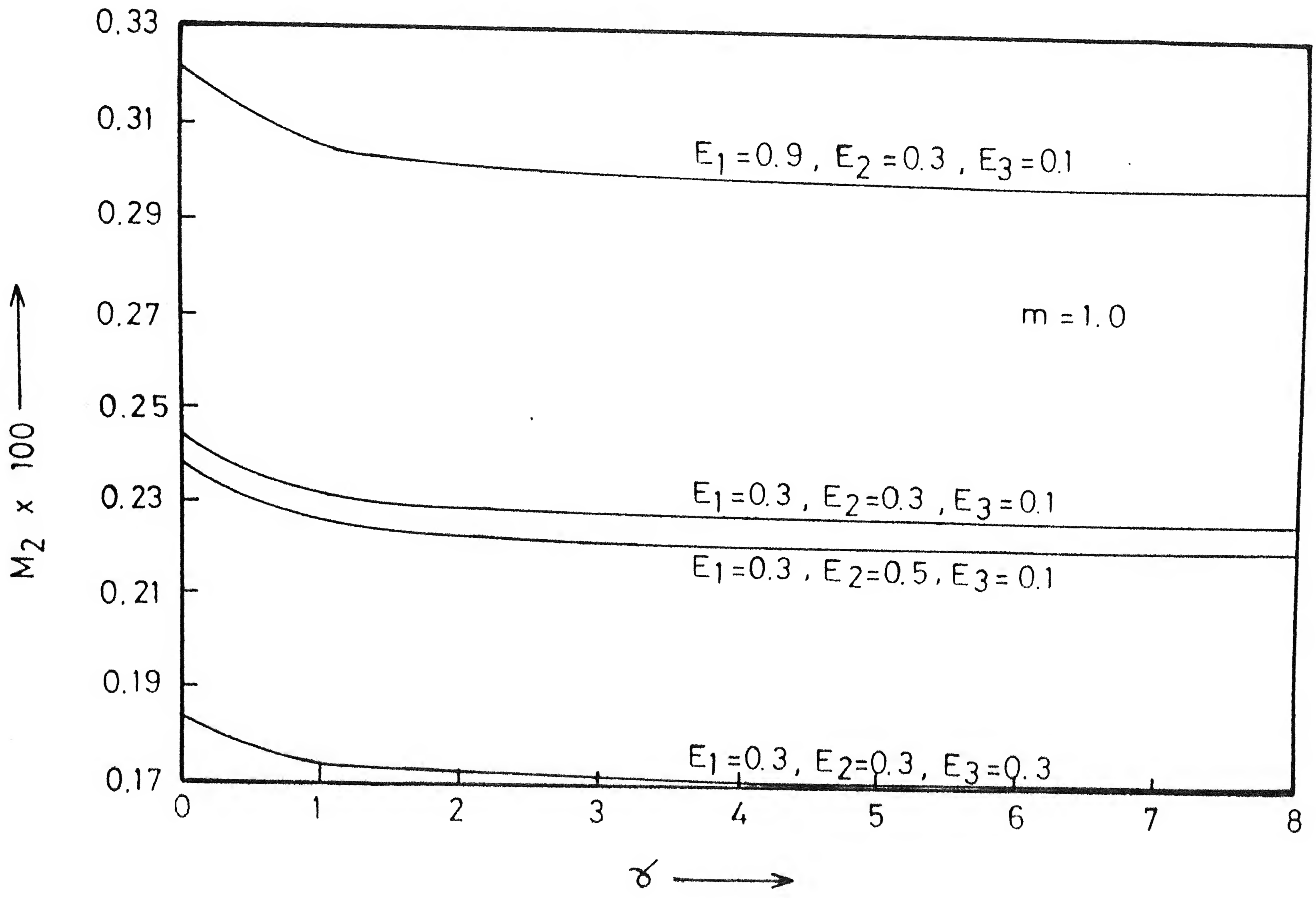


FIG. 3.11 VARIATION OF M_2 WITH γ FOR DIFFERENT E_1, E_2 & E_3.

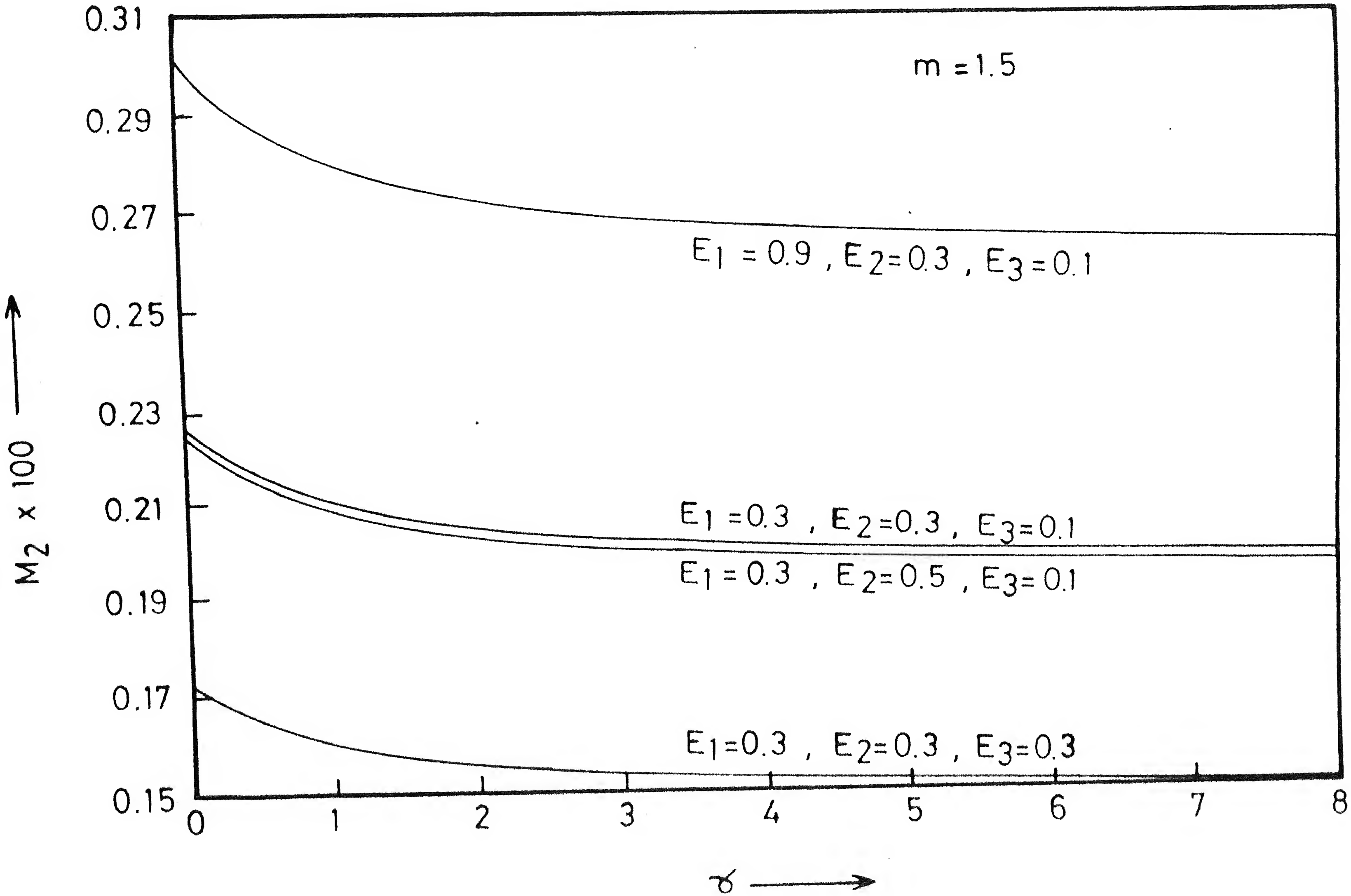


FIG. 3.12 VARIATION OF M_2 WITH γ FOR DIFFERENT E_1 , E_2 & E_3 .

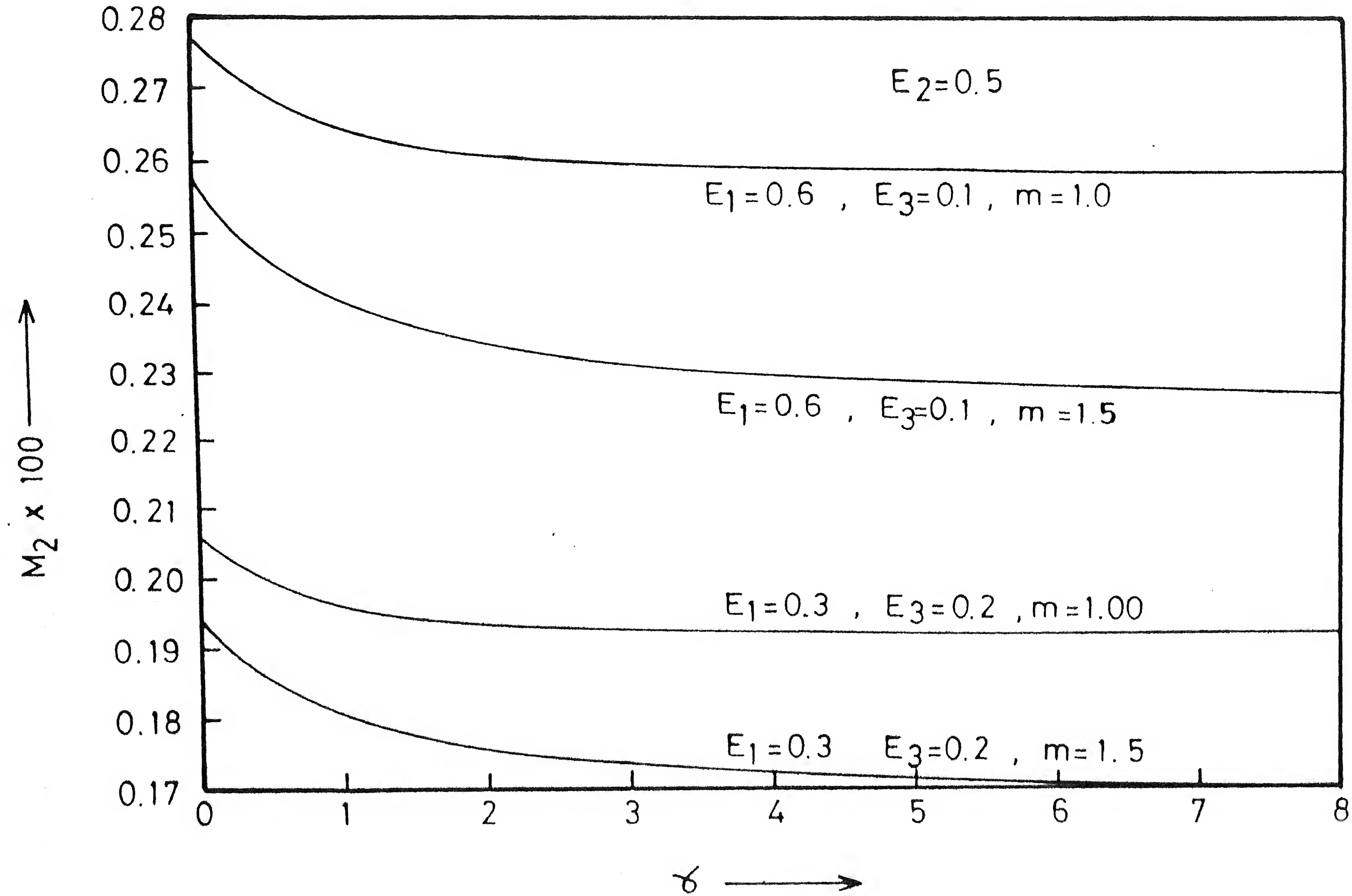


FIG.3.13 VARIATION OF M_2 WITH γ FOR DIFFERENT m, E_1 & E_3.

CHAPTER IV

PERISTALTIC TRANSPORT OF A SIMPLE MICROFLUID

4.1 INTRODUCTION

The study of peristaltic transport of a fluid has received considerable attention during last two decades because of its potential applications in biological systems. The earlier studies were primarily conducted with a view to understand the mechanism of urine transport from kidney to bladder. Therefore, several authors have analysed peristaltic motion of a Newtonian fluid, under various kinds of approximations [Burns and Parkes (1967), Barton and Raynor (1968), Fung and Yih (1969), Chow (1970), Li (1970), Jaffrin and Shapiro (1971), Mittra and Prasad (1974), Manton (1975)]. However, in many other physiological situations the physiological fluid behaves like suspension/non-Newtonian fluid. Hence in recent years attention has been given to study the peristaltic transport of a non-Newtonian fluid in channels/pipes. In particular, many studies have been carried out with reference to vasomotion of the small blood vessels, by considering blood as visco-elastic fluid [Raju and Devanathan (1972), Böhme and Friedrich (1983)], power law fluid [Raju and Devanathan (1972), Radhakrishnamacharya (1982), Shukla et al. (1983)]. Attempts were also made to analyse peristaltic transport of a suspension [Kaimal (1978), Shen et al. (1981), Rath (1983), Srivastava and Srivastava (1989)]. These studies do not consider the particle size effect or rotation and stretchings of the

particles. Renuka Ravindran & Devi (1970), Devi and Devanathan (1975) analysed the problem for micropolar fluids to account for the microrotational effects of the particles, while Srivastava (1986) considered couple stress fluids to see the particle size effect on peristaltic motion.

In this chapter we consider the peristaltic transport of a simple microfluid with sinusoidal wave travelling down the walls of the duct. This chapter consists of two parts. In part I, we have analysed the peristaltic motion of a simple microfluid through a cylindrical tube by using very long wavelength approximation and neglecting the inertial terms [Shapiro et al. (1969), Shukla et al. (1982b)]. In part II, we study the two dimensional flow of a simple microfluid through a channel under peristaltic wave. Approximate solution, upto the first order, is obtained by perturbation analysis with slope parameter as the perturbation parameter [Radhakrishnamacharya (1982)]. The effects of simple microfluid parameters on the pressure rise across the duct have been discussed.

4.2 Part I : FLOW THROUGH A CYLINDRICAL TUBE

We consider here the peristaltic transport of a simple microfluid filled in a cylindrical tube. The walls of the tube are assumed to be executing sinusoidal wave motion due to peristalsis and so the geometry of the tube wall can be described as (Fig. 4.1),

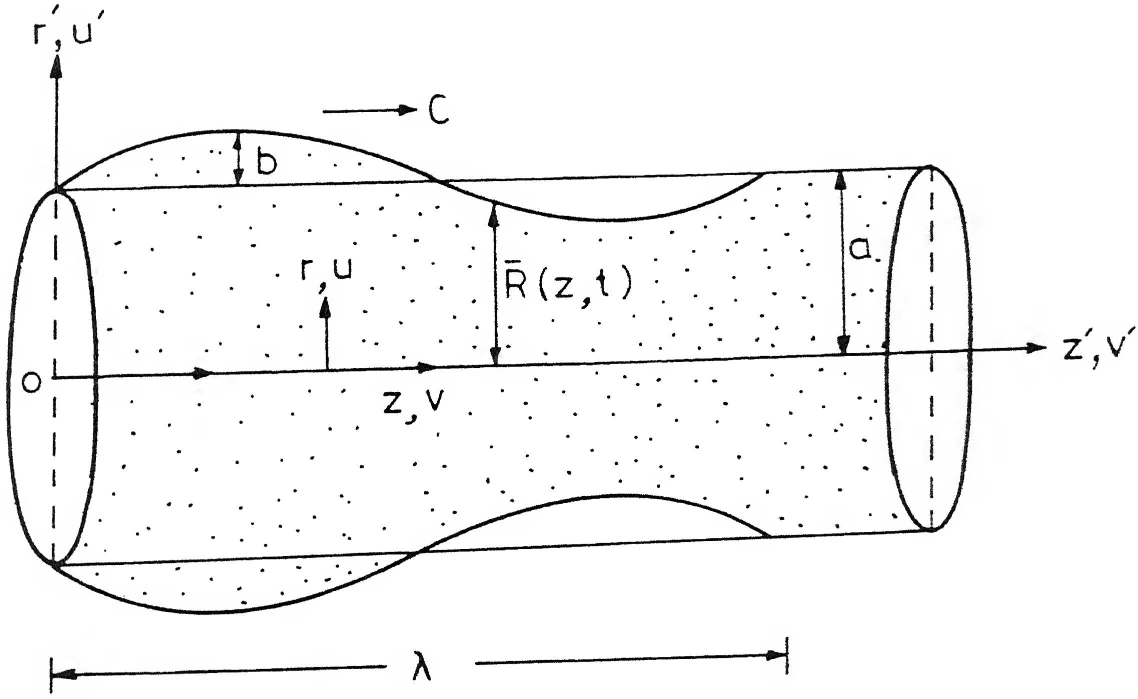


FIG. 4.1 PERISTALTIC TRANSPORT THROUGH A TUBE

$$R = R_o \left\{ 1 + \frac{b}{R_o} \sin \left[\frac{2\pi}{\lambda} (z' - Ct) \right] \right\} \tag{4.1}$$

where R_o, is the radius of the tube in the absence of peristaltic wave, b is the amplitude of the wave, λ the wavelength ,C, the wave propagation velocity, t is the time and z' is the axial co-ordinate.

4.2.1 Formulation

The cylindrical tube is filled with a homogeneous and incompressible simple microfluid and the flow is assumed to be axisymmetric. The constitutive equation and equation of motion are as given in Chapter I. In what follows, we consider a moving frame of reference whose velocity is synchronised with the peristatic wave speed ,C, and moving in the positive axial direction. Thus transforming the stationary co-ordinates (r' and z') to the moving co-ordinates (r,z), we get $r' = r$, $z' = z+Ct$. Further, if (u', o, v') are the velocity components in the stationary co-ordinates and (u, o, v) are those in moving co-ordinates, then, $u' = u$, $v' = v+C$.

Now, using very long wavelength approximation ($R_o \ll \lambda$) and neglecting inertial terms [Shapiro et al. (1969), Shukla et al. (1980e)] equations of motion in the moving co-ordinates reduce to the following dimensionless form

$$- \frac{\partial p}{\partial \bar{r}} = 0 \tag{4.2}$$

$$(1 + \frac{E_3}{2}) \frac{\partial v}{\partial \bar{r}} + E_1 \nu_{(13)} + E_3 \nu_{[13]} = \frac{\bar{r}}{2} \frac{\partial p}{\partial \bar{z}} \tag{4.3}$$

$$\bar{K}_1 \left[\left\{ \nabla^2 - \bar{\alpha}_1^2 \right\} \nu_{31} \right] + \bar{K}_2 \left[\left\{ \nabla^2 - \bar{\alpha}_2^2 \right\} \nu_{13} \right] = 0 \tag{4.4}$$

$$\bar{K}_3 \left[\left\{ \nabla^2 - \bar{\alpha}_3^2 \right\} \nu_{31} \right] + \bar{K}_4 \left[\left\{ \nabla^2 - \bar{\alpha}_4^2 \right\} \nu_{13} \right] = \frac{E_3 \bar{r}}{2+E_3} \frac{\partial p}{\partial \bar{z}} \tag{4.5}$$

where

$$\bar{z} = \frac{z' - Ct}{\lambda} \; ; \quad \bar{t} = \frac{Ct}{\lambda} \quad ; \quad \bar{r} = \frac{r}{R_o}$$

$$\bar{v} = \frac{v'}{C} - 1; \quad \bar{\nu}_{(13)} = \frac{R_o}{C} \nu_{(13)}; \; \bar{\nu}_{[13]} = \frac{R_o}{C} \nu_{[13]} \tag{4.6}$$

$$\bar{p} = \frac{pR_o^2}{\mu C \lambda} \; ; \; \bar{R} = \frac{R}{R_o} = 1 + \varepsilon \sin 2\pi\bar{z}; \; \nabla^2 = \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} - \frac{1}{\bar{r}^2}$$

and $\varepsilon = \frac{b}{R_o} < 1$ is the amplitude ratio, $\bar{K}_i$, E_i as given in Chapter I and II respectively.

[For the sake of simplicity the bars over the flow variables have been omitted here.]

The no-slip condition on velocity component gives,

$$v = -1 \text{ at } \bar{r} = \bar{R}(\bar{z}) \, . \tag{4.7}$$

The boundary conditions on micro-stretch and micro-spin components are assumed as given by Kang and Eringen (1976), i.e.,

$$\nu_{(13)} = A_1 \frac{\partial v}{\partial \bar{r}} \text{ and } \nu_{[13]} = A_2 \frac{\partial v}{\partial \bar{r}} \quad \text{at } \bar{r} = \bar{R}(\bar{z}). \qquad (4.8)$$

The axisymmetry of the flow gives,

$$\frac{\partial v}{\partial \bar{r}} = 0 \; ; \quad \nu_{(13)} = \nu_{[13]} = 0 \qquad \text{at } \bar{r} = 0 . \qquad (4.9)$$

From equation (4.2), it is seen that the pressure p is independent of $\bar{r}$ and it is only weekly dependent on $\bar{t}$ [Shukla et al. (1980)] i.e.

$$p = p(\bar{z},\ \bar{t}) = p(\bar{z}) \qquad (4.10)$$

In the following, we shall write,

$$\frac{\partial p}{\partial \bar{z}} \equiv P .$$

The equations (4.3 to 4.5) are solved along with the boundary conditions, which give the velocity v as,

$$v = -1 + \frac{P}{4}\,(\bar{r}^2-\bar{R}^2) - \frac{P\bar{R}}{4}\left[d_1\left\{I_o(\bar{\alpha}\bar{R})-I_o(\bar{\alpha}\bar{r})\right\} + d_2\left\{I_o(\bar{\beta}\bar{R})-I_o(\bar{\beta}\bar{r})\right\}\right] . \qquad (4.11)$$

[Here d_1 and d_2 are as given in Chapter II with $\bar{R}_1$ replaced by $\bar{R}$]

The flux, q^* in the moving system, is given in dimensionless form by,

$$q = \frac{q^*}{\pi R_o^2 c} = \int_o^{\bar{R}} 2\bar{r}v \, d\bar{r} \;, \tag{4.12}$$

While flux, Q^*, in the stationary frame is given by,

$$Q^* = 2\pi \int_o^{R(z',t)} r'v' \, dr' \;, \tag{4.13}$$

which in dimensionless form Q $(= Q^*/\pi R_o^2 C)$ can be written as

$$Q = 2 \int_o^{\bar{R}(\bar{z})} \bar{r}\,(v+1)\, d\bar{r}$$

$$= q+\bar{R}^2 \quad . \tag{4.14}$$

Using (4.11) in (4.12) we get,

$$q = -\bar{R}^2 - \frac{P\bar{R}^4}{4}\left[\frac{1}{2} + d_1\left\{\frac{\bar{R}^2}{2} I_o(\bar{\alpha}\bar{R}) - \frac{\bar{R}I_1(\bar{\alpha}\bar{R})}{\bar{\alpha}}\right\}\right.$$

$$\left. + d_2\left\{\frac{\bar{R}^2}{2} I_o(\bar{\beta}\bar{R}) - \frac{\bar{R}I_1(\bar{\beta}\bar{R})}{\bar{\beta}}\right\}\right] \tag{4.15}$$

which gives,

$$\frac{dp}{d\bar{z}} = P = \frac{-4}{F(\bar{R})}\left(\frac{q}{\bar{R}^4} + \frac{1}{\bar{R}^2}\right) \tag{4.16}$$

where

$$F(\bar{R}) = \left[\frac{1}{2} + d_1 \left\{ \frac{\bar{R}^2}{2} I_o(\bar{\alpha}\bar{R}) - \frac{\bar{R} I_1(\bar{\alpha}\bar{R})}{\bar{\alpha}} \right\} + d_2 \left\{ \frac{\bar{R}^2}{2} I_o(\bar{\beta}\bar{R}) - \frac{\bar{R} I_1(\bar{\beta}\bar{R})}{\bar{\beta}} \right\} \right] .$$

Further, the time averaged flux $\bar{Q}$, (in dimensionless form), for a complete time period $T = \lambda/c$ is obtained by,

$$\bar{Q} = \int_o^1 Q \, d\bar{t} \tag{4.17}$$

which on using equation (4.14) and the expression for $\bar{R}$ (equation 4.6) gives,

$$\bar{Q} = q+1 + \frac{\epsilon^2}{2} \; . \tag{4.18}$$

The pressure rise,

$$\Delta p = p(0) - p(\lambda) \tag{4.19}$$

across one wavelength is the same whether measured in the fixed or moving co-ordinate system, thus, it can be calculated from the equation (4.16) using the relation :

$$\Delta p = - \int_o^1 \left(\frac{dp}{d\bar{z}} \right) d\bar{z} \; . \tag{4.20}$$

Further, it may be pointed out that, in moving frame of reference, the pressure and flow appear stationary and therefore the flow rate ,q, measured in the moving co-ordinates is a

constant that varies neither with time nor with position along the tube axis [Shapiro et al. (1969)]. Hence, equation (4.16) and (4.20) give,

$$\Delta p = I_1 + qI_2 \tag{4.21}$$

where

$$I_1 = \int_0^1 \frac{4\,dx}{\bar{R}^2 F}, \tag{4.22}$$

and $$I_2 = \int_0^1 \frac{4}{\bar{R}^4 F}\, dx. \tag{4.23}$$

Using equation (4.21) in equation (4.18) we get the expression for $\bar{Q}$ in terms of Δp as,

$$\bar{Q} = \frac{1+\varepsilon^2}{2} + \frac{\Delta p}{I_2} - \frac{I_1}{I_2} \tag{4.24}$$

FRICTION FORCE :

The dimensionless friction force $\bar{\mathfrak{F}}$ $\left(= \frac{F^*}{\pi\lambda C\mu};\right.$ F^* being the friction force at the wall in the stationary co-ordinate system which is same as in the moving system) across one wavelength can be obtained as,

$$\bar{\mathfrak{F}} = \int_0^1 \bar{R}^2 \left(-\frac{dp}{d\bar{z}}\right) d\bar{z}$$

which on subsitution of equation (4.16) gives

$$\bar{\mathcal{F}} = I_3 + qI_1 \tag{4.25}$$

where

$$I_3 = \int_0^1 \frac{4}{\bar{F}}\, dz \; . \tag{4.26}$$

4.2.2 RESULTS AND DISCUSSION

The effect of the flow parameters on pressure rise and friction force is seen through equations (4.21) and (4.25). These equations involve integrals I_1, I_2 and I_3 which are not amenable to analytical solutions and hence have been numerically integrated using Simpson's rule and polynomial approximations for the Bessel functions. The results are presented by taking $\bar{K}_1=1.1$, $\bar{K}_2=1.0$, $\bar{K}_3=-0.9$, $\bar{K}_4=0.6$, $A_1=0.25=A_2$.

In the Figs. (4.2 and 4.3) we see the effect of simple microfluid parameters with $\bar{Q}=0$ (for zero flow) and $\bar{Q}=0.5$ respectively. These figures show that pressure rise ($-\Delta p$) increases as the amplitude ratio (ϵ) increases and the increase is sharp for $\epsilon > 0.3$. Further, the increase in parameter E_1, and decrease in E_3 reduce the pressure rise, while, the parameter E_2 has only a marginal effect on Δp. The reduction in pressure rise due to the variation of the parameters E_1 and E_3 is more in the case of zero flow. Pressure rise gets further reduced as the time averaged flux increases. These effects are elaborated in Figs. (4.4 to 4.6). As expected, the maximum pressure rise occurs at

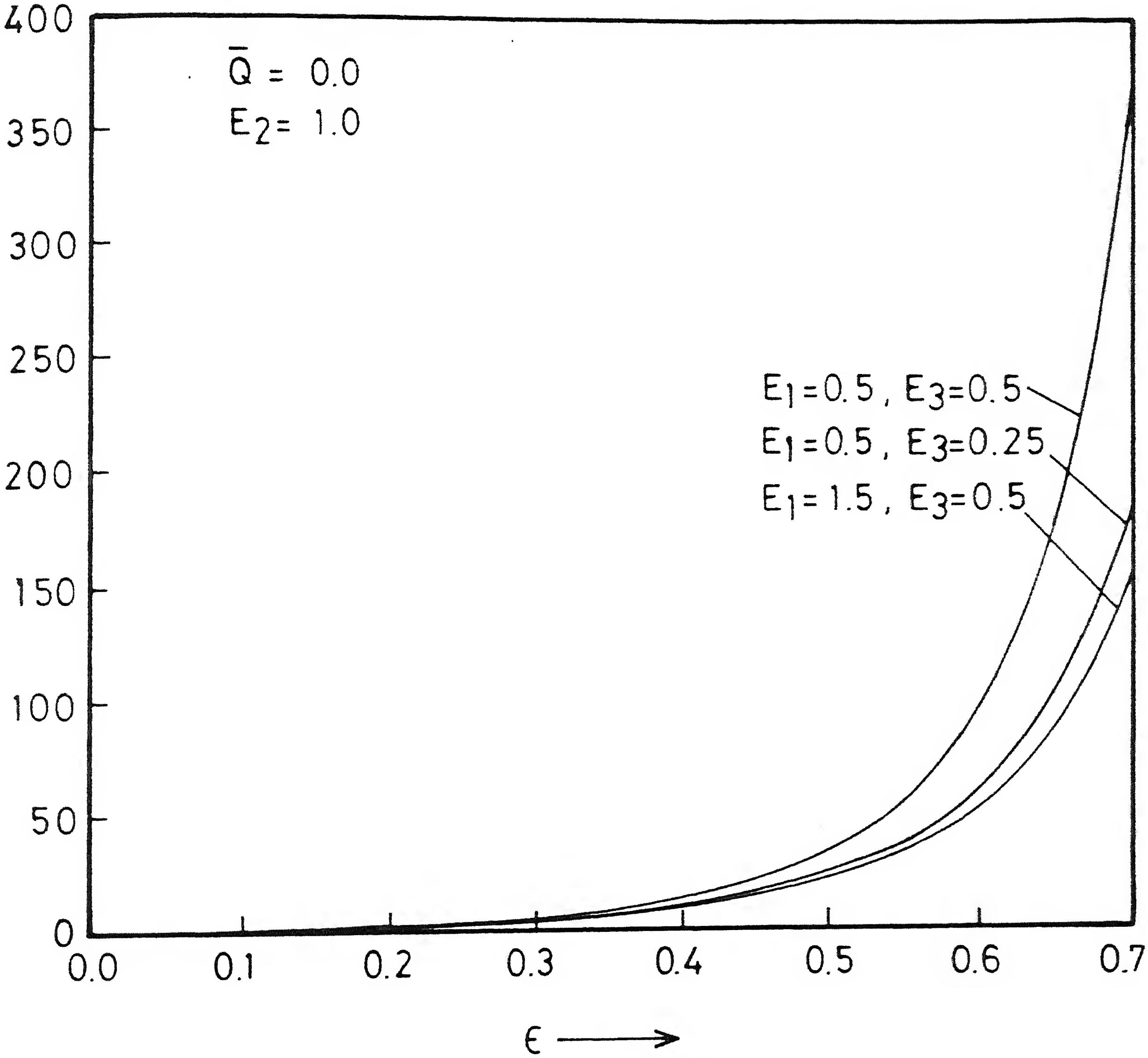


FIG. 4.2 VARIATION OF (-ΔP) WITH ϵ FOR DIFFERENT E_1

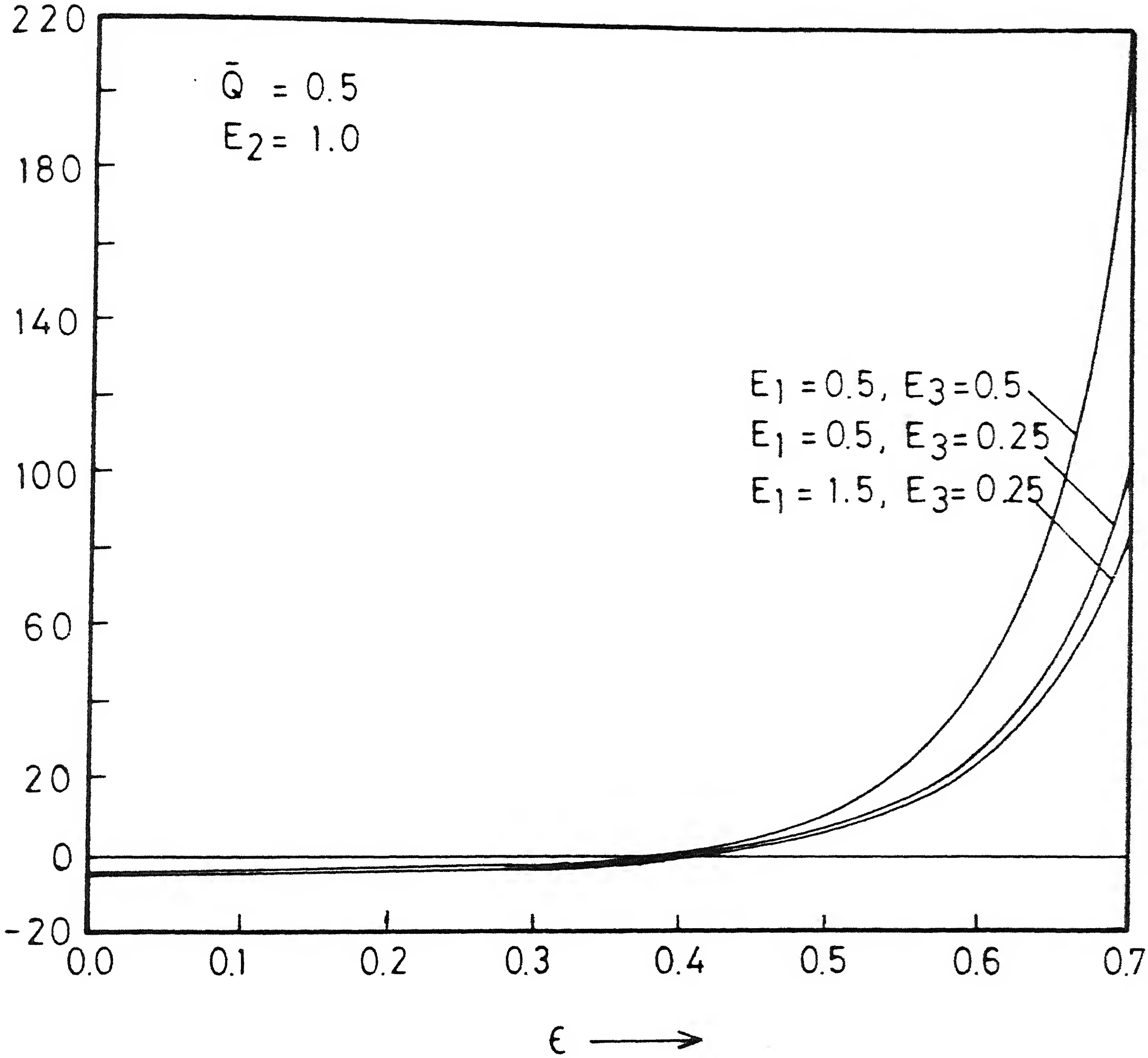


FIG. 4.3 VARIATION OF ($-\Delta P$) WITH ϵ FOR DIFFERENT E_1, E_2 & E_3.

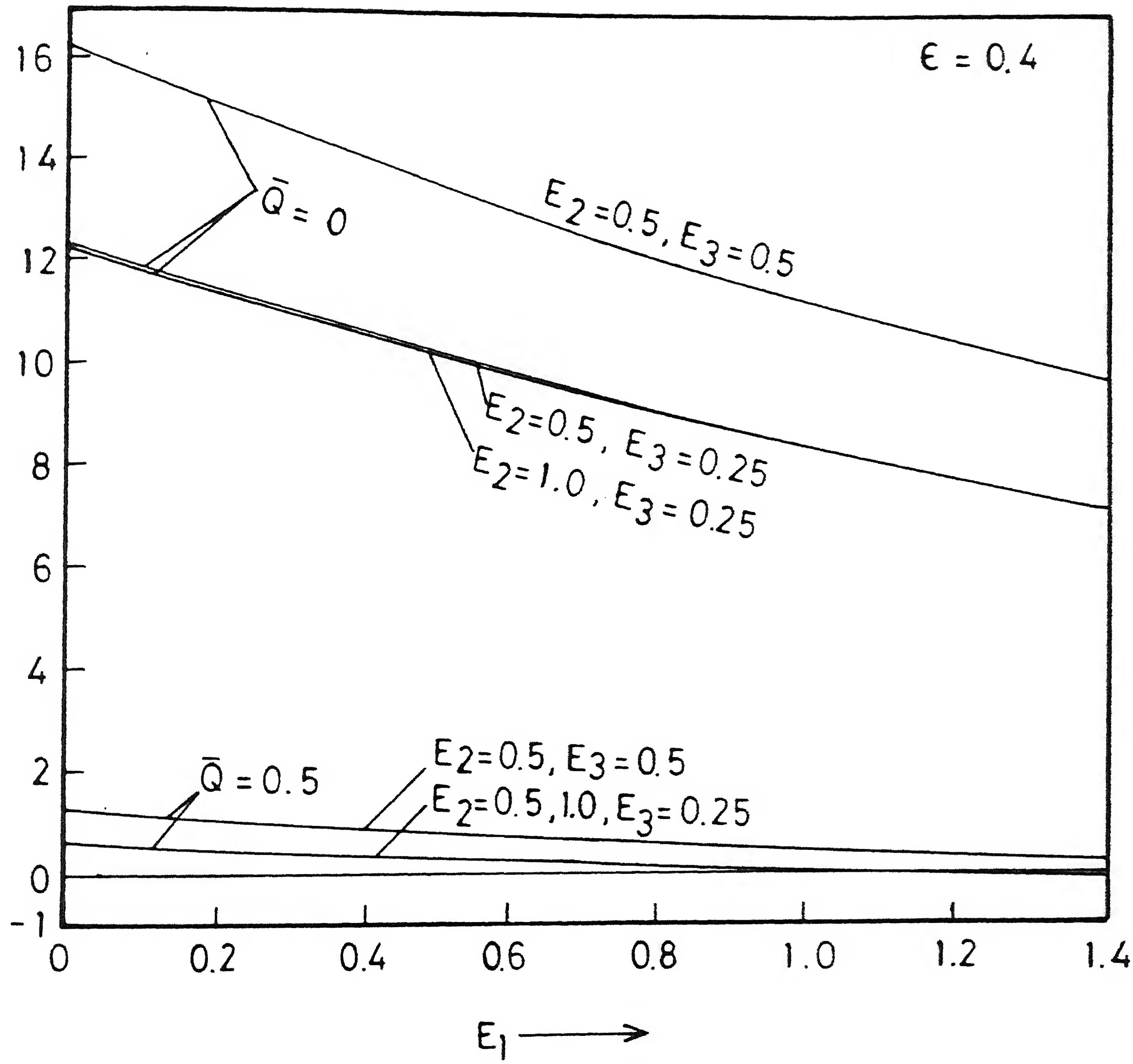


FIG. 4.4 VARIATION OF $(-\Delta P)$ WITH E_1 FOR DIFFERENT E_2, E_3 & $\bar{Q}$.

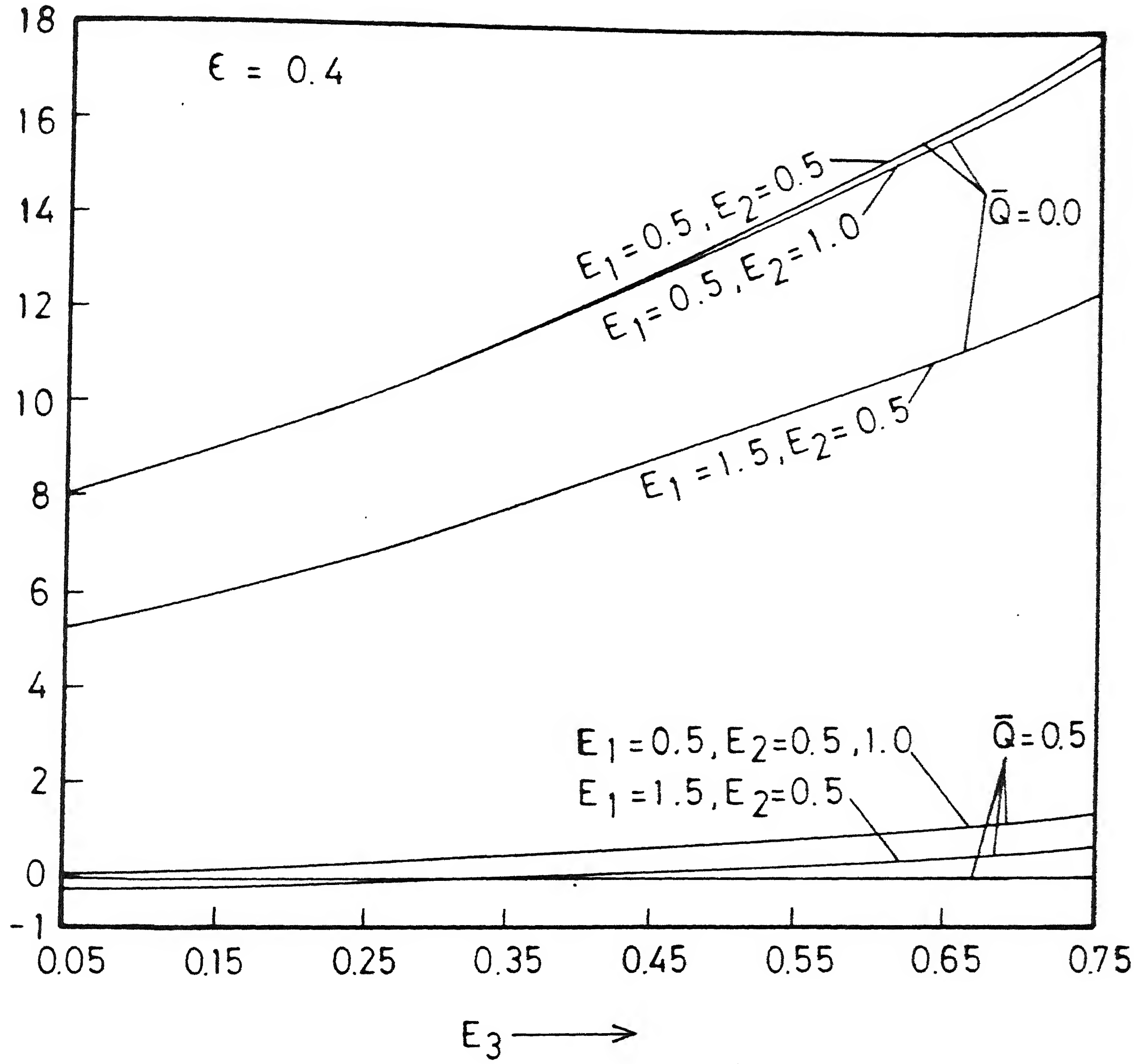


FIG. 4. 5 VARIATION OF $(-\Delta P)$ WITH E_3 FOR DIFFERENT E_1, E_2 & $\bar{Q}$.

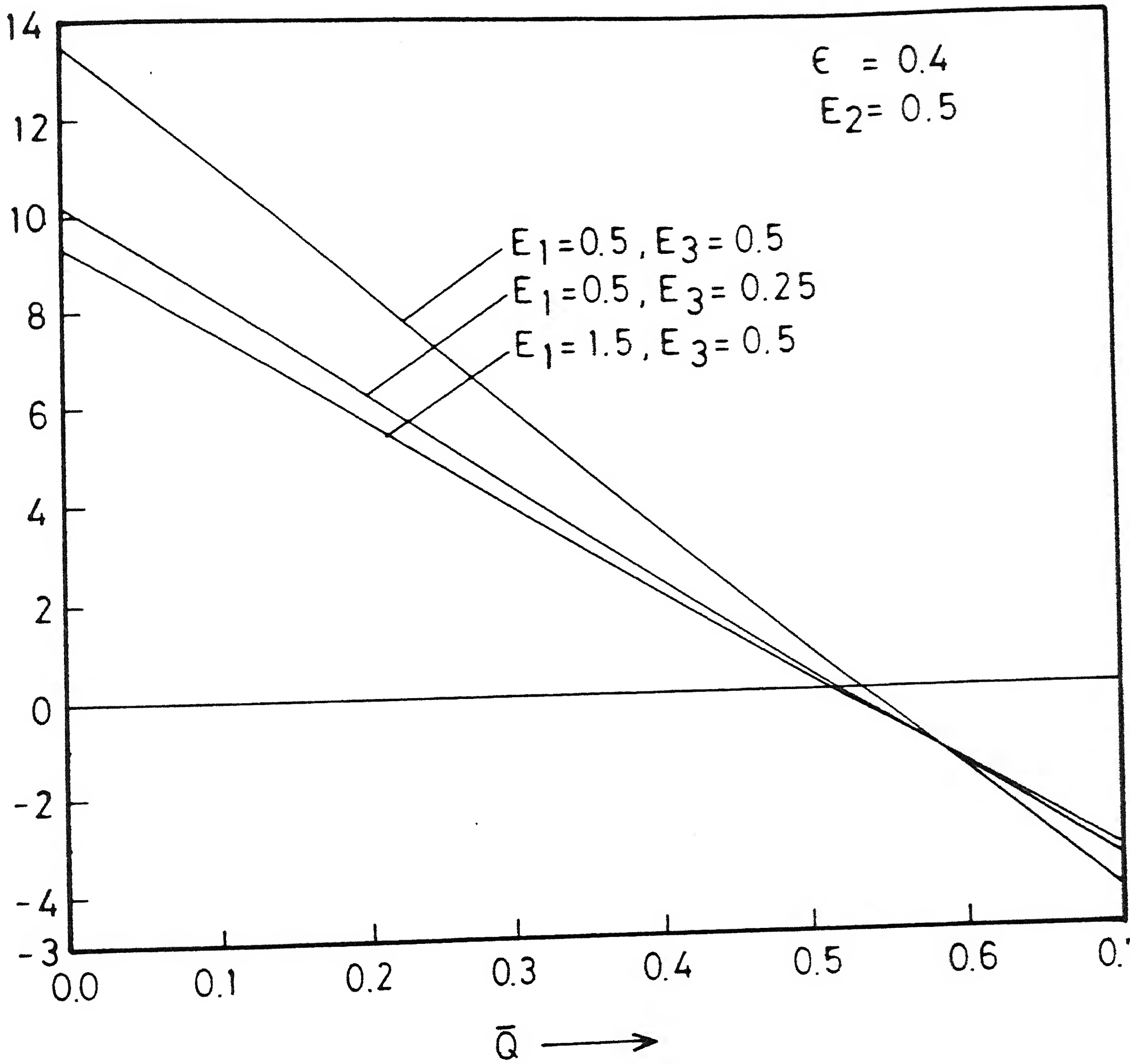


FIG.4.6 VARIATION OF ($-\Delta P$) WITH $\bar{Q}$ FOR DIFFERENT E_1 & E_3.

zero flow rate (Fig. 4.6). Fig. (4.6) also shows that zero pressure difference across a wavelength is achieved around $\bar{Q}=0.5$ and the value of $\bar{Q}$ for which $\Delta p=0$, becomes higher as E_3 increases. The effect of E_1 and E_3 gets reversed for $\bar{Q} > 0.6$.

Friction force at the wall $\bar{\mathcal{F}}$ vs. ϵ (the amplitude ratio) is plotted in Figs. (4.7 and 4.8) while in Figs. (4.9 and 4.10) we have shown its variation with E_3 and $\bar{Q}$ respectively at $\epsilon=0.4$. The effect of various parameters is similar to the effect obtained in the case of pressure drop, however, it is noted that the effect of E_1 and E_3 gets reversed for $\bar{Q} > 0.45$, e.g., Fig. (4.9) shows that the friction force increases with the increase in E_3 at $\bar{Q}=0$ but decreases at $\bar{Q}=0.5$. The friction force $\bar{\mathcal{F}}$ varies linearly with time averaged flux and has almost linear variation with respect to E_3 for $\bar{Q}=0.5$.

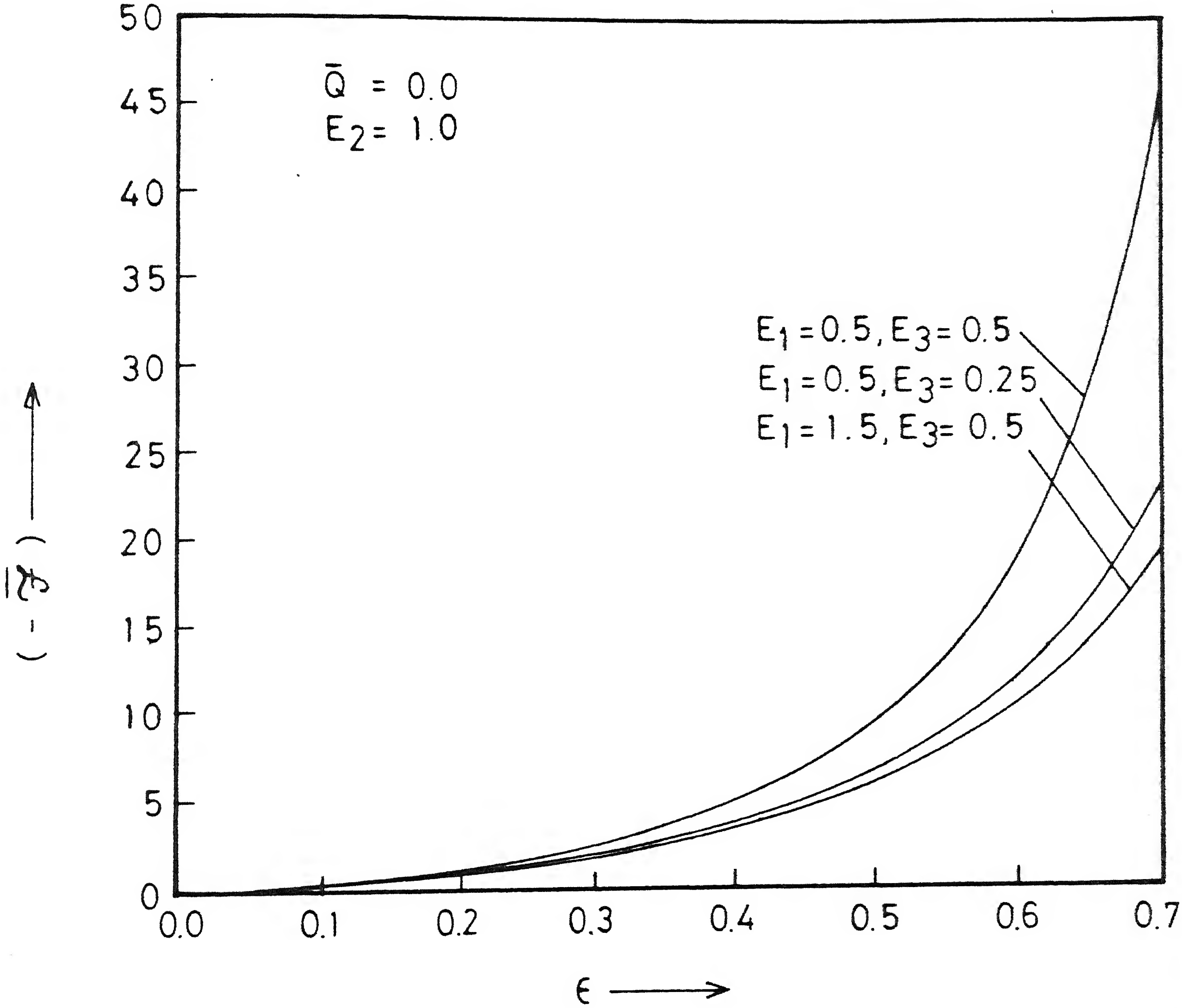


FIG.4.7 VARIATION OF $(-\bar{\mathcal{F}})$ WITH ϵ FOR DIFFERENT E_1 & E_3

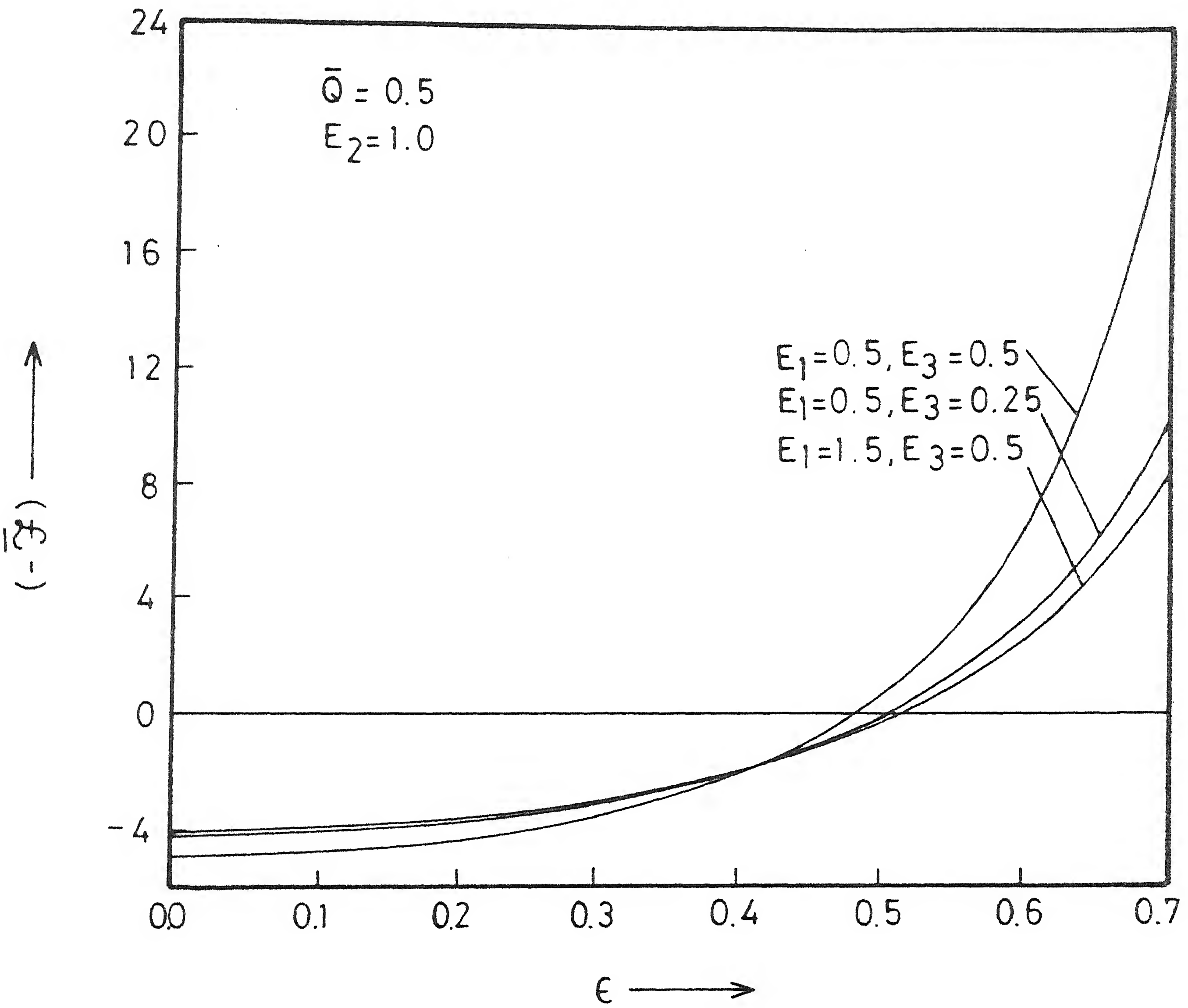


FIG. 4.8 VARIATION OF $(-\bar{\mathcal{F}})$ WITH ϵ FOR DIFFERENT E_1 & E_3.

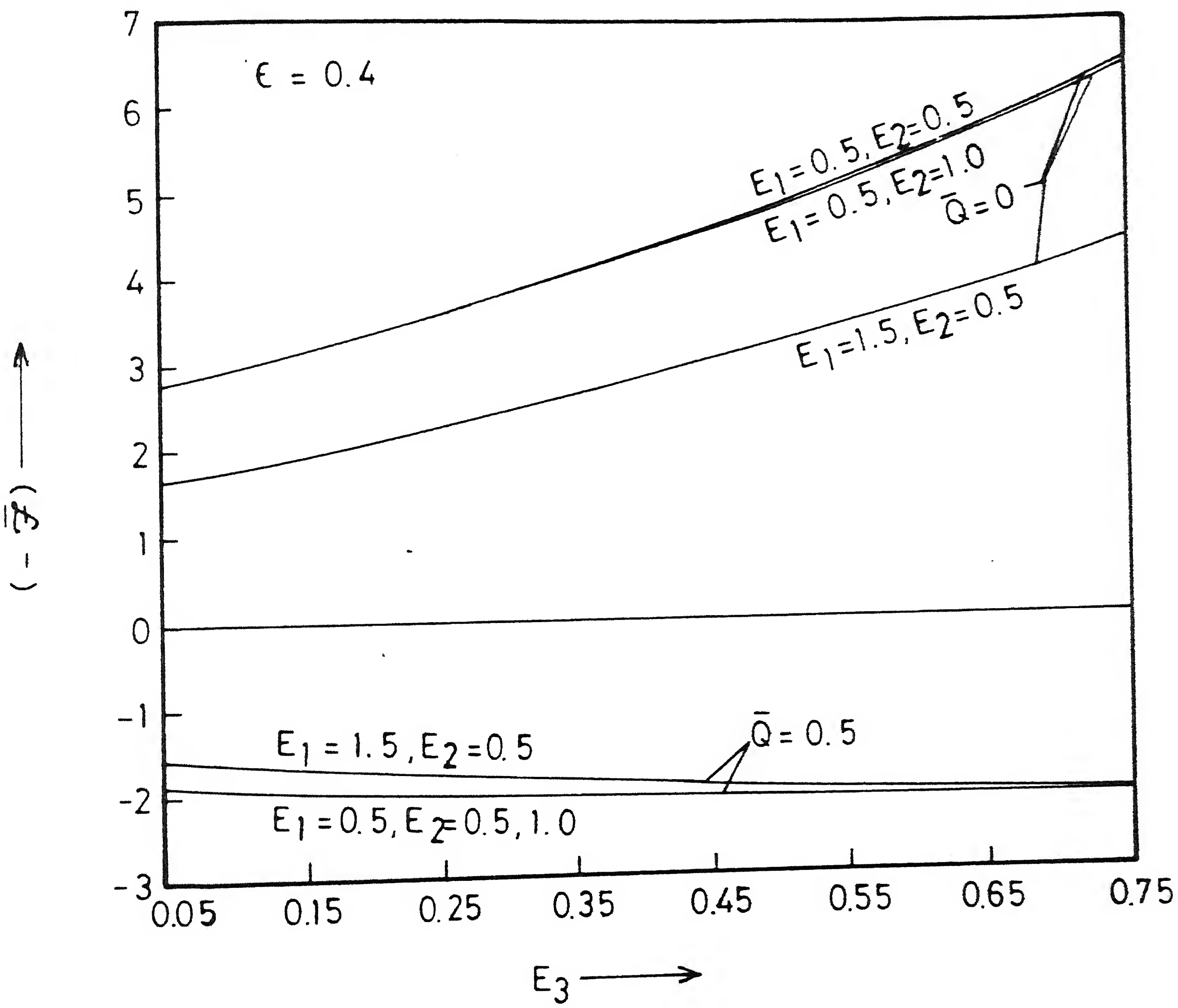


FIG. 4.9 VARIATION OF $(-\bar{\mathcal{F}})$ WITH E_3 FOR DIFFERENT E_1, E_2 & $\bar{Q}$.

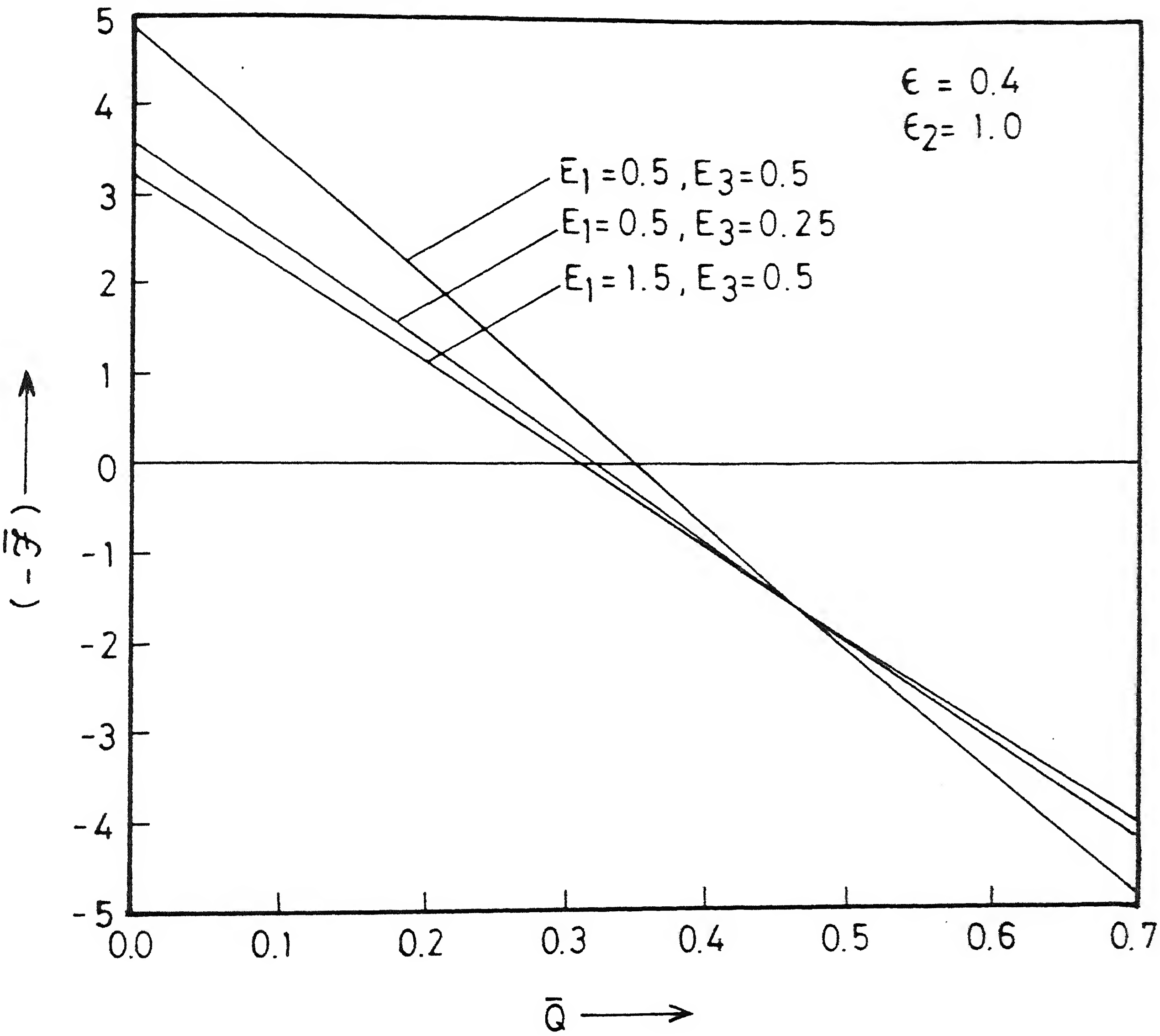


FIG. 4.10 VARIATION OF (- $\bar{\mathcal{F}}$) WITH $\bar{Q}$ FOR DIFFERENT E_1, E_2 & E_3.

4.3 Part II: FLOW THROUGH A CHANNEL

In part I we have studied a simple model for the peristaltic transport of a simple microfluid by considering the flow to be inertia free. In this part we wish to analyse the problem for low Reynolds number flow. It is well known that the problem of peristaltic motion in general involves four parameters i.e. (i) amplitude ratio, (ii) slope parameter, (iii) Reynolds number and (iv) dimensionless time averaged flux or pressure rise per wavelength [Jaffrin and Shapiro (1971)]. It is not possible to get a general analytic solution for arbitrary values of all these parameters, hence, approximate analytical solutions have been attempted by many authors assuming one or more of these parameters to be small, [Burns and Parkes (1967), Barton and Raynor (1968), Fung and Yih (1968), Zien and Ostrach (1970), Devi and Devanathan (1975), Kaimal (1978), Radhakrishnamacharya (1982), etc]. In view of this we consider here peristaltic motion of a simple microfluid through a channel under the assumption that the wavelength (λ) of the peristaltic wave is large in comparison with the mean half width (h) of the channel i.e. the slope parameter ($\varepsilon^* = h/\lambda$) < 1. Thus approximate solutions for a stream function is attempted, using regular perturbation method with ε^* as the perturbation parameter. The analysis is valid for low Reynolds number.

4.3.1 Formulation :

Let us consider the moving frame of reference having velocity of the wave speed C and moving in the positive axial direction. The coordinates in the stationary wave (X', Z') are transformed to those in the moving frame of reference (X, Z) as follows [Fig. 4.1(a)] :

$$X = X' - Ct \quad \text{and} \quad Z = Z' .$$

Thus, the equation of the channel wall $[(Z' = \eta^*(X', t)]$ in the moving frame of reference can be written as,

$$Z = \eta^*(X) = h + b\sin\left(\frac{2\pi X}{\lambda}\right) \tag{4.27}$$

where 2h is the constant mean width of the channel, b is the amplitude and λ is the wavelength of the peristaltic wave.

Now we introduce the following non-dimensional variables. $(u, w) = (U, W)/C$, $\nu_{ij} = \frac{h}{C}\nu'_{ij}$, $p = p'h/\rho C^2\lambda$, $x = \frac{X}{\lambda}$, $z = Z/h$, $\eta = \eta^*/h$, $\varepsilon = b/h$, $\varepsilon^* = h/\lambda$, $R^* = \rho hC/\mu$ is the flow Reynolds number. [where $(U, 0, W)$ are the velocity components in the moving frame of reference and ν'_{ij} - the gyration tensor in the moving coordinate system has only two non-zero components, namely ν'_{13} and ν'_{31}].

Using the above non-dimensional scheme the equations of motion for flow of incompressible simple microfluid in the moving frame of reference can be written as follows :

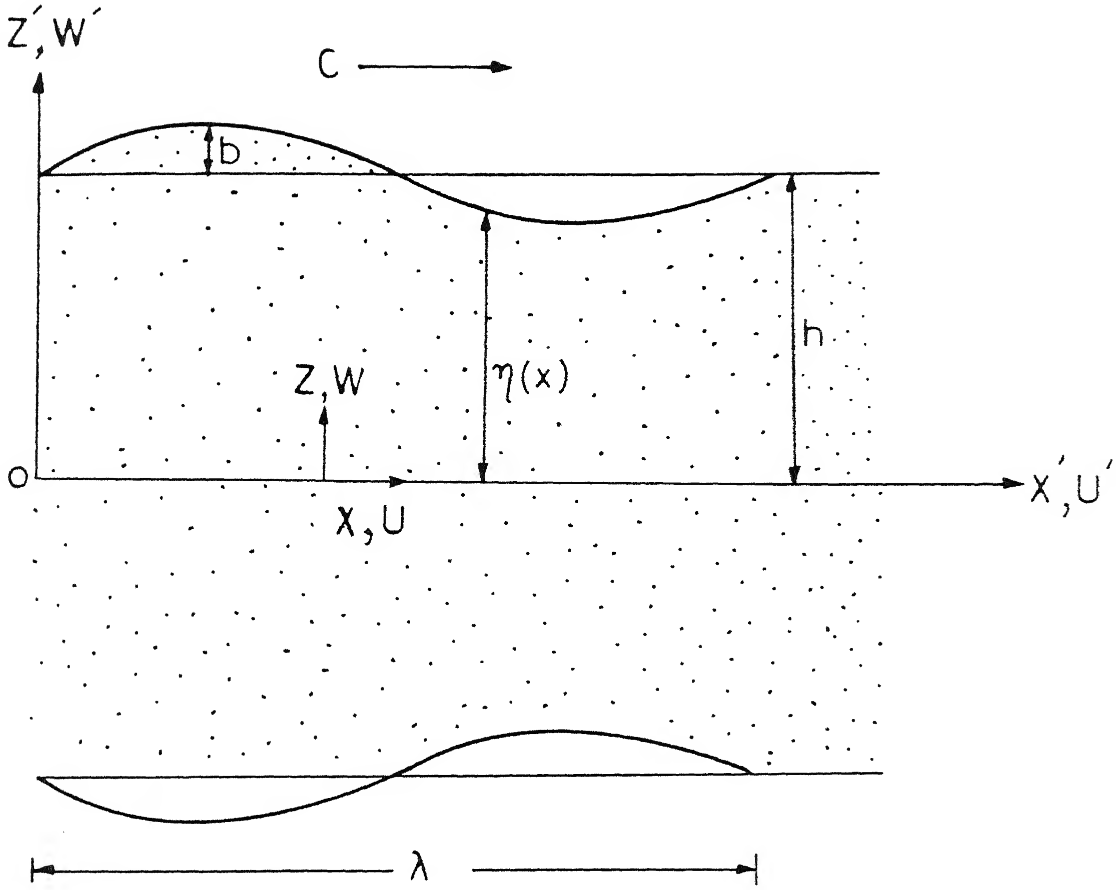


FIG. 4·1(a) PERISTALTIC MOTION IN A CHANNEL

$$-\frac{\partial p}{\partial x} + \left[\varepsilon^* \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]u - E_1\frac{\partial \nu_{(31)}}{\partial z} + E_3\left[\frac{\partial \nu_{[31]}}{\partial z} - \frac{\varepsilon^*}{2}\frac{\partial^2 w}{\partial x \partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}\right]$$

$$= R^*\varepsilon^*\left[u\frac{\partial u}{\partial x} + \varepsilon^* w\frac{\partial u}{\partial z}\right] \qquad (4.28)$$

$$-\varepsilon^*\frac{\partial p}{\partial x} + \left[\varepsilon^{*2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]w - E_1\varepsilon^*\frac{\partial \nu_{(13)}}{\partial x} + E_3\left[\varepsilon^*\frac{\partial \nu_{[13]}}{\partial x} - \frac{\varepsilon^*}{2}\frac{\partial^2 u}{\partial x \partial z} + \frac{\varepsilon^{*2}}{2}\frac{\partial^2 w}{\partial x^2}\right]$$

$$= R^*\varepsilon^*\left[u\frac{\partial w}{\partial x} + \varepsilon^* w\frac{\partial w}{\partial z}\right] \qquad (4.29)$$

$$(\bar{K}_1-\bar{K}_3)\varepsilon^{*2}\frac{\partial^2 \nu_{31}}{\partial x^2} + (\bar{K}_2-\bar{K}_4)\varepsilon^{*2}\frac{\partial^2 \nu_{31}}{\partial x^2} + (\bar{K}_1+\bar{K}_3)\frac{\partial^2 \nu_{13}}{\partial z^2} + (\bar{K}_2+\bar{K}_4)\frac{\partial^2 \nu_{31}}{\partial z^2}$$

$$- 2(E_1+2E_2)\,\nu_{(13)} + E_3\left[2\nu_{[13]} - \frac{\partial u}{\partial z} + \varepsilon^*\frac{\partial w}{\partial x}\right] = 0 \qquad (4.30)$$

$$(\bar{K}_1+\bar{K}_3)\varepsilon^{*2}\frac{\partial^2 \nu_{31}}{\partial x^2} + (\bar{K}_2+\bar{K}_4)\varepsilon^{*2}\frac{\partial^2 \nu_{31}}{\partial x^2} + (\bar{K}_1-\bar{K}_3)\frac{\partial^2 \nu_{13}}{\partial z^2} + (\bar{K}_2-\bar{K}_4)\frac{\partial^2 \nu_{31}}{\partial z^2}$$

$$- 2(E_1+2E_2)\,\nu_{(31)} + E_3\left[2\nu_{[31]} - \varepsilon^*\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] = 0 \qquad (4.31)$$

alongwith the equations of conservation of mass i.e. $u_{kk} = 0$, $\nu_{kk} = 0$ (where E_1, E_2, E_3, $\bar{K}_i$ are as defined in Chapter I & II).

The non-dimensional boundary conditions in moving frame of reference can be written as :

At the channel wall, we have no slip condition for velocity component, where as the gyration tensor is assumed to satisfy the condition as given by eqn. (1.25) in Chapter I.

$$\left.\begin{aligned}
&\text{(i)} && u = -1 \\
&\text{(ii)} && \nu_{(13)} = A_1 d_{13} = \frac{A_1}{2}\left[\frac{\partial u}{\partial z} + \varepsilon^* \frac{\partial w}{\partial x}\right] \\
&\text{(iii)} && \nu_{[13]} = A_2 w_{13} = \frac{A_2}{2}\left[\frac{\partial u}{\partial z} - \varepsilon^* \frac{\partial w}{\partial x}\right]
\end{aligned}\right\} z=\eta(x) \qquad (4.32)$$

$$\left.\begin{aligned}
&\text{(iv)} && w = 0, \quad \frac{\partial u}{\partial z} = 0 \\
&\text{(v)} && \nu_{13} = \nu_{31} = 0
\end{aligned}\right\} \text{at } z = 0 . \qquad (4.33)$$

Further we prescribe flux ($2Q^*$) in the moving coordinates. This in the non-dimensional form gives,

$$\int_0^{\eta(x)} u \, dz = q \qquad \text{(which is constant)}$$

[Note that the flux $2Q^*$ in the moving coordinates is related to $2Q$ in stationary system by $Q = Q^* + \eta Ch$ and is constant].

Now introducing the stream function ψ

$$u = \frac{\partial \psi}{\partial z} \qquad \text{and} \qquad w = -\varepsilon^* \frac{\partial \psi}{\partial x}$$

the equations (4.28 to 4.33) can be rewritten in terms of stream function after eliminating p, as

$$R^{*}\varepsilon^{*}\left[\frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\left[\frac{\partial^2\psi}{\partial z^2}+\varepsilon^{*2}\frac{\partial^2\psi}{\partial x^2}\right]-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\left[\frac{\partial^2\psi}{\partial z^2}+\varepsilon^{*2}\frac{\partial^2\psi}{\partial x^2}\right]\right]$$

$$-\varepsilon^{*2}\left[\frac{\partial^4\psi}{\partial z^2\partial x^2}\right](2+E_3)-\left[\frac{\partial^4\psi}{\partial z^4}+\varepsilon^{*4}\frac{\partial^4\psi}{\partial x^4}\right]\left[1+\frac{E_3}{2}\right]$$

$$+E_1\left[\frac{\partial^2\nu_{(31)}}{\partial z^2}-\varepsilon^{*2}\frac{\partial^2\nu_{(13)}}{\partial x^2}\right]-E_3\left[\frac{\partial^2\nu_{[31]}}{\partial z^2}-\varepsilon^{*2}\frac{\partial^2\nu_{[13]}}{\partial z^2}\right]=0$$

(4.34)

$$(\bar{K}_1-\bar{K}_3)\varepsilon^{*2}\frac{\partial^2\nu_{31}}{\partial x^2}+(\bar{K}_2-\bar{K}_4)\varepsilon^{*2}\frac{\partial^2\nu_{13}}{\partial x^2}+(\bar{K}_1+\bar{K}_3)\frac{\partial^2\nu_{13}}{\partial z^2}+(\bar{K}_2+\bar{K}_4)\frac{\partial^2\nu_{31}}{\partial z^2}$$

$$-2(E_1+2E_2)\,\nu_{(13)}+E_3\left[2\nu_{[13]}-\frac{\partial^2\psi}{\partial z^2}-\varepsilon^{*2}\frac{\partial^2\psi}{\partial x^2}\right]=0 \quad (4.35)$$

$$(\bar{K}_1+\bar{K}_3)\varepsilon^{*2}\frac{\partial^2\nu_{31}}{\partial x^2}+(\bar{K}_2+\bar{K}_4)\varepsilon^{*2}\frac{\partial^2\nu_{13}}{\partial x^2}+(\bar{K}_1-\bar{K}_3)\frac{\partial^2\nu_{13}}{\partial z^2}+(\bar{K}_2-\bar{K}_4)\frac{\partial^2\nu_{31}}{\partial z^2}$$

$$-2(E_1+2E_2)\,\nu_{(31)}+E_3\left[2\nu_{[31]}+\varepsilon^{*2}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right]=0 \quad (4.36)$$

The corresponding boundary conditions are

$$\left.\begin{aligned}\frac{\partial^2\psi}{\partial z^2}&=0\\ \psi&=0\\ \nu_{13}&=\nu_{31}=0\end{aligned}\right\}\ \text{at}\quad z=0 \qquad (4.37)$$

$$\frac{\partial \psi}{\partial z} = -1 \qquad \text{at} \quad z = \eta$$

$$\left.\begin{aligned} \nu_{(13)} &= A_1 \left[\frac{\partial^2 \psi}{\partial z^2} - \varepsilon^{*2} \frac{\partial^2 \psi}{\partial x^2} \right] \qquad \text{at } z = \eta \\ \nu_{[13]} &= A_2 \left[\frac{\partial^2 \psi}{\partial z^2} + \varepsilon^{*2} \frac{\partial^2 \psi}{\partial x^2} \right] \qquad \text{at } z = \eta \end{aligned}\right\} \tag{4.38}$$

The condition prescribing flux in the moving frame yields,

$$\psi = q \qquad \text{at} \qquad z = \eta \,. \tag{4.39}$$

In the following, we attempt an approximate solution of the above system using the perturbation method, with perturbation parameter being the slope parameter ε^*. Thus

$$\left.\begin{aligned} \psi &= \psi_o + \varepsilon^* \psi_1 + 0\,(\varepsilon^{*2}) \\ \nu_{13} &= M_o + \varepsilon^* M_1 + 0\,(\varepsilon^{*2}) \\ \nu_{31} &= N_o + \varepsilon^* N_1 + 0\,(\varepsilon^{*2}) \end{aligned}\right\} \tag{4.40}$$

Using the above expressions in eqns. (4.34 to 4.36) and assuming that $R^* \varepsilon^* \underset{\sim}{} 0\,(\varepsilon^*)$, we collect the coefficients of like powers of ε^*, which give the coupled equations governing ψ, M and N of different orders as follows.

ZEROTH ORDER

$$(2+E_3)\frac{\partial^4\psi_o}{\partial z^4} - E_1\frac{\partial^2}{\partial z^2}(M_o+N_o) - E_3\frac{\partial^2}{\partial z^2}(M_o-N_o) = 0 \tag{4.41}$$

$$(\bar{K}_1-\bar{K}_3)\frac{\partial^2 M_o}{\partial z^2} +(\bar{K}_2-\bar{K}_4)\frac{\partial^2 N_o}{\partial z^2} -(E_1+2E_2)(M_o+N_o)+E_3\left[(N_o-M_o)+\frac{\partial^2\psi_o}{\partial z^2}\right]=0 \tag{4.42}$$

$$(\bar{K}_1+\bar{K}_3)\frac{\partial^2 M_o}{\partial z^2} +(\bar{K}_2+\bar{K}_4)\frac{\partial^2 N_o}{\partial z^2} -(E_1+2E_2)(M_o+N_o)+E_3\left[(M_o-N_o)-\frac{\partial^2\psi_o}{\partial z^2}\right]=0 \tag{4.43}$$

With corresponding boundary condtlions

$$\left.\begin{array}{llll}
\psi_o & = 0 & \text{at} & z = 0 \\
\dfrac{\partial\psi_o}{\partial z} & = -1 & \text{at} & z = \eta \\
\psi_o & = q & \text{at} & z = \eta \\
\nu^o_{(13)} & = \frac{1}{2}(M_o+N_o) = A_1\dfrac{\partial^2\psi_o}{\partial z^2} & \text{at} & z = \eta \\
\nu^o_{[13]} & = \frac{1}{2}(M_o-N_o) = A_2\dfrac{\partial^2\psi_o}{\partial z^2} & \text{at} & z = \eta
\end{array}\right\} \tag{4.44}$$

FIRST ORDER

$$(2+E_3)\frac{\partial^4\psi_1}{\partial z^4} -E_1\frac{\partial^2}{\partial z^2}(M_1+N_1)-E_3\frac{\partial^2}{\partial z^2}(M_1-N_1) = 2R^*\left[\frac{\partial\psi_o}{\partial z}\frac{\partial^3\psi_o}{\partial x\partial z^2} - \frac{\partial\psi_o}{\partial x}\frac{\partial^3\psi_o}{\partial z^3}\right] \tag{4.45}$$

$$(\bar{K}_1+\bar{K}_3)\frac{\partial^2 M_1}{\partial z^2} +(\bar{K}_2+\bar{K}_4)\frac{\partial^2 N_1}{\partial z^2} -(E_1+2E_2)(M_1+N_1)+E_3\left[(M_1-N_1) - \frac{\partial^2 \psi_1}{\partial z^2}\right] = 0 \tag{4.46}$$

$$(\bar{K}_1-\bar{K}_3)\frac{\partial^2 M_1}{\partial z^2} +(\bar{K}_2-\bar{K}_4)\frac{\partial^2 N_1}{\partial z^2} -(E_1+2E_2)(M_1+N_1)-E_3\left[(M_1-N_1) + \frac{\partial^2 \psi_1}{\partial z^2}\right] = 0 \tag{4.47}$$

The corresponding boundary condtiions are

$$\left.\begin{array}{llll}
\psi_1 & = 0 & \text{at} & z = 0 \\
\dfrac{\partial \psi_1}{\partial z} & = 0 & \text{at} & z = \eta \\
\psi_1 & = 0 & \text{at} & z = \eta \\
\nu^1_{(13)} & = \dfrac{(M_1+N_1)}{2} & = A_1 \dfrac{\partial^2 \psi_1}{\partial z^2} & \text{at } z = \eta \\
\nu^1_{[13]} & = \dfrac{(M_1-N_1)}{2} & = A_2 \dfrac{\partial^2 \psi_1}{\partial z^2} & \text{at } z = \eta
\end{array}\right\} \tag{4.48}$$

Integrating eqn. (4.41) twice and using the condition at z = 0, we get,

$$\frac{\partial^2 \psi_o}{\partial z^2} = \frac{1}{(2+E_3)}\left[E_1(M_o+N_o) + E_3(M_o-N_o) + 2Az\right] \tag{4.49}$$

(where A is the integration constant).

The eqns. (4.42) and (4.43) on using above equation can be written in the following form :

$$\bar{K}_1 \left[\frac{d^2}{dz^2} - \alpha_1^2 \right] M_o + \bar{K}_2 \left[\frac{d^2}{dz^2} - \alpha_2^2 \right] N_o = 0 \tag{4.50}$$

$$\bar{K}_3 \left[\frac{d^2}{dz^2} - \alpha_3^2 \right] M_o - \bar{K}_4 \left[\frac{d^2}{dz^2} - \alpha_4^2 \right] N_o = \frac{AE_3 z}{(2+E_3)} \tag{4.51}$$

(where α_i are as defined in Chapter III). Now equations (4.50) and (4.51) can be solved in the usual way as in Chapter III. Thus solving for M_o, N_o and ψ_o we get,

$$\psi_o = \sum_{i=1}^{2} \left\{ F_i(z) + \frac{1}{2\eta^3} F_i(\eta)\,(z^3 - 3\eta^2 z) \right\} G_i - z - \frac{(q+\eta)}{2\eta^3}(z^3 - 3\eta^2 z) \tag{4.52}$$

$$M_o = \sum_{i=1}^{2} \left\{ (\frac{3P_1}{\eta^3} F_i(\eta) z - P_2(a_i^2 - \alpha_5^2) \sinh a_i z \; G_i(\eta) \right\} - \frac{3P_1 (q+\eta) z}{\eta^3} \tag{4.53}$$

$$N_o = \sum_{i=1}^{2} \left\{ \left[\sinh a_i z - \frac{3P_1 F_i(\eta) z}{\eta^3} \right] G_i(\eta) \right\} + \frac{3P_1}{\eta^3}(q+\eta) z \tag{4.54}$$

where $F_i(z) = \left[\frac{\sinh a_i z}{a_i} - z \cosh a_i \eta \right] \cdot \frac{e_i}{a_i}$

$$G_1(\eta) = 3(q+\eta)\left[P_6(1-2A_1-2A_2)+P_4(1+2A_1-2A_2)\right] \;/\; \left[2\eta^2(P_3P_6-P_4P_5)\right]$$

$$G_2(\eta) = -3(q+\eta)\left[P_5(1-2A_1-2A_2)+P_3(1+2A_2-2A_3)\right] \;/\; \left[2\eta^2(P_3P_6-P_4P_5)\right]$$

$$P_1 = 2\alpha_1^2 E_3 K_1 \Big/ \left[\alpha^2\beta^2(2+E_3)(K_1K_4 - K_2K_3)\right]$$

$$P_2 = (K_1K_4 - K_2K_3) \Big/ \left[K_1K_3(\alpha_1^2 - \alpha_3^2)\right]$$

$$P_{2+i}(\eta) = 3\left[\frac{1}{2} - A_1 - A_2\right]\frac{F_i(\eta)}{\eta^2} - \left[\sinh a_i\eta\right]\left[(\alpha_i^2-\alpha_5^2)P_2+(A_1+A_2)e_i^*\right] \tag{4.55}$$

$$P_{4+i}(\eta) = -3\left[\frac{1}{2} + A_1 + A_2\right]F_i(\eta) + \left[(1 - A_1 + A_2)e_i^*\right]\sinh a_i\eta$$

e_i^* $(i=1,2)$ are as defined in Chapter III (Eqn. 3.16) and

$$\alpha_5^2 = \frac{\bar{K}_1\bar{K}_4\,\alpha_4^2 - \bar{K}_1\bar{K}_3\,\alpha_2^2}{(\bar{K}_1\bar{K}_4 - \bar{K}_2\bar{K}_3)}$$

[Note : Throughout this chapter the summation index ,i, takes the value 1 and 2]. Also $a_1 = \bar{\alpha}$, $a_2 = \bar{\beta}$ and $\bar{\alpha}^2$, $\bar{\beta}^2$ are the roots of the eqn. (2.38).

Using eqn. (4.52), we find ψ_{oz}, ψ_{ox}, ψ_{oxzz}, ψ_{ozzz}, which gives :

$$\begin{aligned}(\psi_{oz}\psi_{xzzz} - \psi_{ox}\psi_{ozzz}) &= z\left[q^2P_7+qP_8+P_9\right] + z^3\left[q^2P_{10}+qP_{11}+P_{12}\right] \\ &-\sum_i\Big[\sinh(a_iz)\left(q^2P_{i3} + qP_{i4} + P_{i5}\right) + \cosh(a_iz)\left(q^2P_{i6} + qP_{i7} + P_{i8}\right) \\ &+ z^2\sinh(a_iz)\left(q^2Q_{i1} + qQ_{i2} + Q_{i3}\right) - z^3\cosh(a_iz)\left(q^2Q_{i4} + qQ_{i5} + Q_{i6}\right) \\ &- (-1)^{i+1}\sinh(a_iz)\cosh(a_{i+1}z)\left(q^2Q_{i7}+qQ_{i8}+Q_{i9}\right)\Big]\end{aligned} \tag{4.56}$$

where P_1 to P_{12} and P_{ij}, Q_{ij} are known functions of η and η'. Integrating equation (4.45) twice with respect to z and using the boundary conditions at z=0, we get

$$(2+E_3)\frac{\partial^2\psi_1}{\partial z^2} - \frac{E_1}{(2+E_3)}(M_1+N_1) - \frac{E_3}{(2+E_3)}(M_1-N_1) = B(z) \tag{4.57}$$

$$B(z) = \frac{2}{2+E_3}\left[R\int_0^z\int_0^{z^*}\left[\frac{\partial\psi_o}{\partial\bar{z}}\frac{\partial^3\psi_o}{\partial x\partial\bar{z}^2} - \frac{\partial\psi_o}{\partial x}\frac{\partial^3\psi_o}{\partial\bar{z}^3}\right](d\bar{z})(dz^*)+Jz\right] \tag{4.58}$$

where J is integration constant which is to be found out later.

Equations (4.46) and (4.47) along with equations (4.57) give :

$$M_1 = \frac{-E_3B(z)}{\bar{K}_3(\alpha_3^2-\alpha_1^2)} - \frac{(\bar{K}_3\bar{K}_2 - \bar{K}_1\bar{K}_4)}{\bar{K}_1\bar{K}_3\ (\alpha_3^2-\alpha_1^2)}\left[D^2-\alpha_5^2\right]N_1 \tag{4.59}$$

Also we get,

$$(D^2-\bar{\alpha}^2)(D^2-\bar{\beta}^2)N_1 = \left[D^2-\alpha_1^2\right]\left[\frac{E_3B(\bar{z})\ K_1}{(\bar{K}_1\bar{K}_4-\bar{K}_2\bar{K}_3)}\right] \tag{4.60}$$

Solving for M_1 and N_1 using the boundary conditions $M_1=N_1=0$ at z=0, we get

$$M_1+N_1 = \frac{-E_3B(z)}{K_3(\alpha_3^2-\alpha_1^2)} + P_{30}Jz + (q^2P_{31} + qP_{32} + P_{33})\ z$$

$$+ (q^2P_{34} + qP_{35} + qP_{36})\ z^3 + (q^2P_{37} + qP_{38} + P_{39})\ z^5$$

$$- A_3 \sum_i \sinh(a_i z)(q^2 U_{i1} + qU_{i2} + U_{i3}) + \sum_i \Big[z \cosh(a_i z)(q^2 W_{i4} + qW_{i5} + W_{i6})$$

$$+ z^2 \sinh(a_i z)\ (q^2 W_{i7} + qW_{i8} + W_{i9}) + z^3 \cosh(a_i z)\ (q^2 X_{i1} + qX_{i2} + X_{i3})$$

$$+ z^4 \sinh(a_i z)\ (q^2 X_{i4} + qX_{i5} + X_{i6}) + \sinh(a_i + a_{i+1})z\ (q^2 X_{i7} + qX_{i8} + X_{i9})$$

$$+ \sinh(a_i - a_{i+1})z(q^2 Y_{i1} + qY_{i2} + Y_{i3}) \Big] + M\Big[- A_3(\bar{\alpha}^2 - \alpha_5^2) + 1 \Big] \sinh(\bar{\alpha} z)$$

$$+ P \Big[- A_3\ (\bar{\beta}^2 - \alpha_5^2) + 1 \Big] \sinh\ (\bar{\beta} z) \tag{4.61}$$

$$M_1 - N_1 = \frac{-E_3 B(z)}{K_3(\alpha_3^2 - \alpha_1^2)} + P_{40} J z + (q^2 P_{41} + qP_{42} + P_{43})\ z$$

$$+ (q^2 P_{44} + qP_{45} + qP_{46})\ z^3 + (q^2 P_{47} + qP_{48} + P_{49})\ z^5$$

$$- A_3 \sum_i \sinh(a_i z)(q^2 U_{i1} + qU_{i2} + U_{i3}) + \sum_i \Big[z \cosh(a_i z)(q^2 Y_{i4} + qY_{i5} + Y_{i6})$$

$$+ z^2 \sinh(a_i z)\ (q^2 Y_{i7} + qY_{i8} + Y_{i9}) + z^3 \cos Z(a_i z)\ (q^2 Z_{i1} + qZ_{i2} + Z_{i3})$$

$$+ z^4 \sinh(a_i z)\ (q^2 Z_{i4} + qZ_{i5} + Z_{i6}) + \sinh(a_i + a_{i+1})z\ (q^2 Z_{i7} + qZ_{i8} + Z_{i9})$$

$$+ \sinh(a_i - a_{i+1})z(q^2 A_{i1} + qA_{i2} + A_{i3}) \Big] + M\Big[- A_3(\bar{\alpha}^2 - \alpha_5^2) - 1 \Big] \sinh(\bar{\alpha} z)$$

$$+ P \Big[- A_3\ (\bar{\beta}^2 - \alpha_5^2) - 1 \Big] \sinh\ (\bar{\beta} z) \tag{4.62}$$

where $A_3 = \dfrac{(\bar{K}_3\bar{K}_2 - \bar{K}_1\bar{K}_4)}{\bar{K}_1\bar{K}_3\ (\alpha_3^2-\alpha_1^2)}$,

M & P are constants of integration and are yet to be determined, while A_{ij}, U_{ij}, W_{ij}, X_{ij}, Y_{ij}, Z_{ij} are known functions of η and its derivative.

Substituting M_1 and N_1 from eqns. (4.61), (4.62) in eqn. (4.57) and integrating the resultant equation with the boundary conditions

$$\frac{\partial\psi_1}{\partial z} = 0 \text{ at } z = \eta \quad \text{and} \quad \psi_1 = 0 \text{ at } z = 0,$$

we get,

$$\psi_1 = P_{50}\, J\left(\frac{z^3}{6} - \frac{\eta^2 z}{2}\right) + (q^2P_{51} + qP_{52} + P_{53})\left(\frac{z^3}{6} - \frac{\eta^2 z}{2}\right)$$

$$+ (q^2P_{54}+qP_{55}+P_{56})\left(\frac{z^5}{6} - \frac{\eta^4 z}{4}\right) + (q^2P_{57}+qP_{58}+P_{59})\left(\frac{z^7}{42} - \frac{\eta^6 z}{6}\right)$$

$$+ (q^2P_{57} + qP_{58} + P_{59})\left(\frac{z^7}{42} - \frac{\eta^6 z}{6}\right)$$

$$+ \sum_i \left[\frac{\sinh(a_i z)}{a_i^2} - \frac{z\cosh(a_i z)}{a_i}\right]\left[q^2 E_{i1} + q\, E_{i2} + E_{i3}\right]$$

$$+ \sum_i \left[\frac{z\cosh(a_i z)}{a_i^2} - \frac{2\sinh(a_i z)}{a_i^3} - \frac{z\eta\sinh(a_i\eta)}{a_i} + \frac{z\cosh(a_i\eta)}{a_i^2}\right]$$

$$\left[q^2 E_{i4} + q\, E_{i5} + E_{i6}\right]$$

$$
+ \left[\frac{z^2 \sinh(a_i z)}{a_i{}^2} - \frac{4z\cosh(a_i z)}{a_i^3} + \frac{6\sinh(a_i z)}{a_i^4} - \frac{\eta^2 z\cosh(a_i z)}{a_i} \right.
$$

$$
\left. + \frac{2\eta z\sinh(a_i\eta}{a_i^2} - \frac{2z\cosh(a_i\eta)}{a_i^3} \right] \left[q^2 E_{i7} + qE_{i8} + E_{i9} \right]
$$

$$
+ \sum_i \left[\frac{z^3\cosh(a_i z)}{a_i^2} - \frac{6z^2\sinh(a_i z)}{a_i^3} + \frac{18z\cosh(a_i z)}{a_i^4} - \frac{24\sinh(a_i z)}{a_i^5} \right.
$$

$$
\left. - \frac{\eta^3 z\sinh(a_i\eta)}{a_i} + \frac{3\eta^2 z\cosh(a_i\eta)}{a_i^2} - \frac{6\eta z\sinh(a_i\eta)}{a_i^3} + \frac{6z\cosh(a_i z)}{a_i^4} \right]
$$

$$
\left[q^2 F_{i1} + qF_{i2} + F_{i3} \right]
$$

$$
+ \sum_i \left[\frac{z^4\sinh(a_i z)}{a_i^2} - \frac{8z^3\cosh(a_i z)}{a_i^3} + \frac{36z^2\sinh(a_i z)}{a_i^4} - \frac{96z\cosh(a_i z)}{a_i^5} \right.
$$

$$
+ \frac{120\sinh(a_i z)}{a_i^6} - \frac{z\eta^4\cosh(a_i\eta)}{a_i} + \frac{4\eta^3 z\sinh(a_i\eta)}{a_i^2} - \frac{12\eta^2 z\cosh(a_i\eta)}{a_i^3}
$$

$$
\left. + \frac{24\eta z\sinh(a_i\eta)}{a_i^4} - \frac{24z\cosh(a_i\eta)}{a_i^5} \right] \left[q^2 B_{i7} + qB_{i8} + B_{i9} \right]
$$

$$
+ \sum_i \left[\frac{\sinh(a_i + a_{i+1})}{(a_i + a_{i+1})^2} - z\,\frac{\cosh(a_i + a_{i+1})\eta}{(a_i + a_{i+1})} \right] \left[q^2 F_{i4} + qF_{i5} + F_{i6} \right]
$$

$$+ \sum_i \left[\frac{\sinh(a_i - a_{i+1})}{(a_i - a_{i+1})^2} - z \frac{\cosh(a_i - a_{i+1})\eta}{(a_i - a_{i+1})} \right] \left[q^2 F_{i7} + qF_{i8} + F_{i9} \right]$$

$$+ M\ P_{60} \left[\frac{\sinh(\bar{\alpha} z)}{\bar{\alpha}^2} - \frac{z \cosh(\bar{\alpha}\eta)}{\bar{\alpha}} \right]$$

$$+ P\ P_{61} \left[\frac{\sinh(\bar{\beta} z)}{\bar{\beta}^2} - \frac{z \cosh(\bar{\beta}\eta)}{\bar{\beta}} \right] \tag{4.63}$$

The remaining boundary conditions at $z = \eta(x)$ determine P, M and J as follows :

$$P = J\ P_{47} + q^2 P_{75} + qP_{76} + P_{77} + P_{78} \sum_i (q^2 A^*_{i1} + qA^*_{i2} + A^*_{i3})$$

$$- P_{79} \sum_i (q^2 A^*_{i4} + qA^*_{i5} + A^*_{i6}) \tag{4.64}$$

$$M = J\ P_{80} + q^2 P_{81} + qP_{82} + P_{83} + P_{84} \sum_i (q^2 A^*_{i1} + qA^*_{i2} + A^*_{i3})$$

$$+ P_{85} \sum_i (q^2 A^*_{i4} + qA^*_{i5} + A^*_{i6}) \tag{4.65}$$

$$J = P_{86}\ (q^2\ P_{87} + q\ P_{88} + P_{89})$$

$$+ \sum_i \sinh(a_i \eta) \left[q^2\ A^*_{i7} + q\ A^*_{i8} + A^*_{i9} \right]$$

$$+ \sum \cosh(a_i \eta) \left[q^2\ B^*_{i1} + q\ B^*_{i2} + B^*_{i3} \right]$$

$$+ \sum \sinh(a_i + a_{i+1})\eta \left[q^2\ B^*_{i4} + q\ B^*_{i5} + B^*_{i6} \right]$$

$$+ \sum \cosh(a_i + a_{i+1})\eta \left[q^2 B^*_{i7} + q B^*_{i8} + B^*_{i9} \right]$$

$$+ \sum \sinh(a_i - a_{i+1})\eta \left[q^2 C^*_{i1} + q C^*_{i2} + C^*_{i3} \right]$$

$$+ \sum \cosh(a_i - a_{i+1})\eta \left[q^2 C^*_{i4} + q C^*_{i5} + C^*_{i6} \right]$$

$$+ \sum (q^2 A^*_{i1} + qA^*_{i2} + A^*_{i3}) \left[P_{60} P_{84} \left\{ \frac{\sinh(\bar{\alpha}\eta)}{\bar{\alpha}^2} - \frac{\eta\cosh(\bar{\alpha}\eta)}{\bar{\alpha}} \right\} \right.$$

$$\left. + P_{61} P_{78} \left\{ \frac{\sinh(\bar{\beta}\eta)}{\bar{\beta}^2} - \frac{\eta\cosh(\bar{\beta}\eta)}{\bar{\beta}} \right\} \right]$$

$$+ \sum_i (q^2 A^*_{i4} + qA^*_{i5} + A^*_{i6}) \left[P_{60} P_{85} \left\{ \frac{\sinh(\bar{\alpha}\eta)}{\bar{\alpha}^2} - \frac{\eta\cosh(\bar{\alpha}\eta)}{\bar{\alpha}} \right\} \right.$$

$$\left. - P_{61} P_{90} \left\{ \frac{\sinh(\bar{\beta}\eta)}{\bar{\beta}^2} - \frac{\eta\cosh(\bar{\beta}\eta)}{\bar{\beta}} \right\} \right] \qquad (4.66)$$

STRESS AT THE WALL

The stress at the wall is given by

$$\tau = \frac{t_{13} \left\{ 1 - \left(\frac{d\eta^*}{dX}\right)^2 \right\} + \left\{ t_{33} - t_{11} \right\} \frac{d\eta^*}{dX}}{1 + \left(\frac{d\eta^*}{dX}\right)^2} \qquad (4.67)$$

From the eqn. (4.67) after omitting the second and higher order powers of ε^*, we get the stress at the wall as

$$\tau \simeq t_{13} + (t_{33} - t_{11}) \frac{d\eta^*}{dX} \quad \text{at} \quad Z = \eta^*(X). \qquad (4.68)$$

Thus, we get

$$(\frac{h}{\mu C})t_{13} = \left[\left(\frac{\partial^2\psi}{\partial z^2} - \varepsilon_1^2\frac{\partial^2\psi}{\partial x^2}\right) - E_1\nu_{(13)} + E_3\left(\nu_{[13]} - \frac{1}{2}\frac{\partial^2\psi}{\partial z^2} - \varepsilon_1^2\frac{\partial^2\psi}{\partial x^2}\right)\right] \tag{4.69}$$

$$(\frac{h}{\mu C})\left[t_{13} - t_{11}\right] = -4\varepsilon^*\frac{\partial^2\psi}{\partial z\partial x}$$

Substituting these value in eqn. (4.68) we get

$$\bar{\tau}_w = \frac{\tau h}{\mu C} = \left[(1 - \frac{E_3}{2} - E_1A_1 + E_3A_2\right]\left[\frac{\partial^2\psi_o}{\partial z^2} + \varepsilon_1\frac{\partial^2\psi_1}{\partial z^2}\right] \quad \text{at } z = \eta(x) \tag{4.70}$$

This value is numerically calculated for different set of parameters of simple microfluid using the expressions for ψ_o and ψ_1.

PRESSURE DROP

Eqn. (4.20) in terms of stream function can be written as

$$-\frac{\partial p}{\partial x} + \frac{\partial^3\psi}{\partial z^3} - \frac{E_1}{2}\frac{\partial}{\partial z}\left[M_o+N_o + \varepsilon_1(M_1+N_1)\right] - \frac{E_3}{2}\frac{\partial}{\partial z}\left[M_o-N_o+\varepsilon^*(M_1-N_1)\right]$$
$$+\frac{E_3}{2}\frac{\partial^3\psi}{\partial z^3} = R^*\varepsilon^*\left[\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2}\right]. \tag{4.71}$$

which gives the equations governing zeroth and first order pressure as follows :

$$\frac{\partial p_o}{\partial x} = \frac{E_3}{2}\frac{\partial}{\partial z}\left[N_o - M_o\right] + (1 + \frac{E_3}{2})\frac{\partial^3 \psi_o}{\partial z^3} - \frac{E_1}{2}\frac{\partial}{\partial z}(M_o + N_o) \qquad (4.72)$$

$$\frac{\partial p_1}{\partial x} = \frac{E_3}{2}\frac{\partial}{\partial z}\left[M_1 - N_1\right] + (1 + \frac{E_3}{2})\frac{\partial^3 \psi_1}{\partial z^3} - \frac{E_1}{2}\frac{\partial}{\partial z}(M_1 + N_1)$$

$$- R^*\left[\psi_{oz}\,\psi_{oxz} - \psi_{ox}\,\psi_{ozz}\right] \qquad (4.73)$$

Therefore, we get

$$\frac{\partial p}{\partial x} = \frac{\partial p_o}{\partial x} + \varepsilon^* \frac{\partial p_1}{\partial x}$$

$$= q^2\left[(\eta)'\varepsilon^* P_{91}\left\{3P_2P_5 - P_1P_5 - \frac{1}{\eta^6}\left[\sum_i \frac{(G_i)' e_i^2 G_i}{\alpha_i^2}\right] + \frac{3}{\eta^7}\sum_i \frac{G_i^2 e_i^2}{\alpha_i^2}\right\}\right.$$

$$+ q\left[\varepsilon^* P_{92} - \frac{3}{\eta^3}\left[\sum_i \frac{F_i(\eta)G_i}{\eta^3} + 1\right] + \varepsilon^* R^*\left\{(\eta)'(P_1 - 2\eta P_1 P_5 + 5P_2P_5\eta)\right.\right.$$

$$\left.\left. - \sum\left[(G_i)' - \frac{G_i}{\eta^2}\right]\frac{2e_i^2 G_i(\eta)'}{\eta^5 a_i^2} + \sum_i\left[\frac{3e_i^2 G_i^2(\eta)'}{\eta^6 a_i^2}\right] - (\eta)' 3P_2\right\}\right]$$

$$+ \left[\varepsilon^* P_{93} - \frac{3}{\eta^3}\left[\sum_i \frac{F_i(\eta)G_i}{\eta^2} + \eta\right] + \varepsilon^* R^*\left\{\eta\left[(\eta)' P_1 - P_1P_5\eta(\eta)'\right.\right.\right.$$

$$\left. + 2P_2P_5\eta(\eta)'\right] - \left[(G_1)' - \frac{2}{\eta}G_1\right]\frac{e_1^2 G_1(\eta)'}{\eta^4\alpha^2} - \left[(G_2)' - G_2\right]\frac{e_2^2 G_2(\eta)'}{\eta^2\beta^2}$$

$$\left.\left.\left. 2\eta(\eta)' P_2\right\}\right]\right] + O(\varepsilon^{*2}) \qquad (4.74)$$

The pressure drop across the wavelength thus, can be evaluated by

$$\Delta p = - \int_0^1 \left(\frac{dp}{dx} \right) dx \qquad (4.75)$$

Using eqn.(4.75) Δp is also numerically calculated and its values are plotted for different values of the simple microfluid parameters.

4.3.2 RESULTS AND DISCUSSION

We have presented approximate solution for a stream function as an asymptotic series in terms of slope parameter ϵ^*. It may be pointed out that the zeroth order solution ψ_o gives the limiting solution in the limit $\epsilon^* \to 0$ which corresponds to the situation of very long wavelength. Further in order to discuss the results qualitatively the expressions for pressure rise($-\Delta p$) and stress at the wall i.e. $\bar{\tau}_w$ have been calculated numerically taking $\bar{K}_1$=1.1, $\bar{K}_2$=1.0, $\bar{K}_3$=-0.9, $\bar{K}_4$=0.6, A_1=0.25, A_2=0.25 and q=0.2. The results are plotted in Figs.(4.11 to 4.20) for various values of different parameters.

In Figs. (4.11 to 4.15) the effect of the simple microfluid parameter and the Reynolds number is shown. As in the case of part I, pressure rise increases with the amplitude ratio and this increase becomes very sharp after ϵ=0.4,, when R^*=1.0. Also pressure rise increases as the parameter E_3 increases, and it reduces with the increase in the parameters E_1 and E_2. The

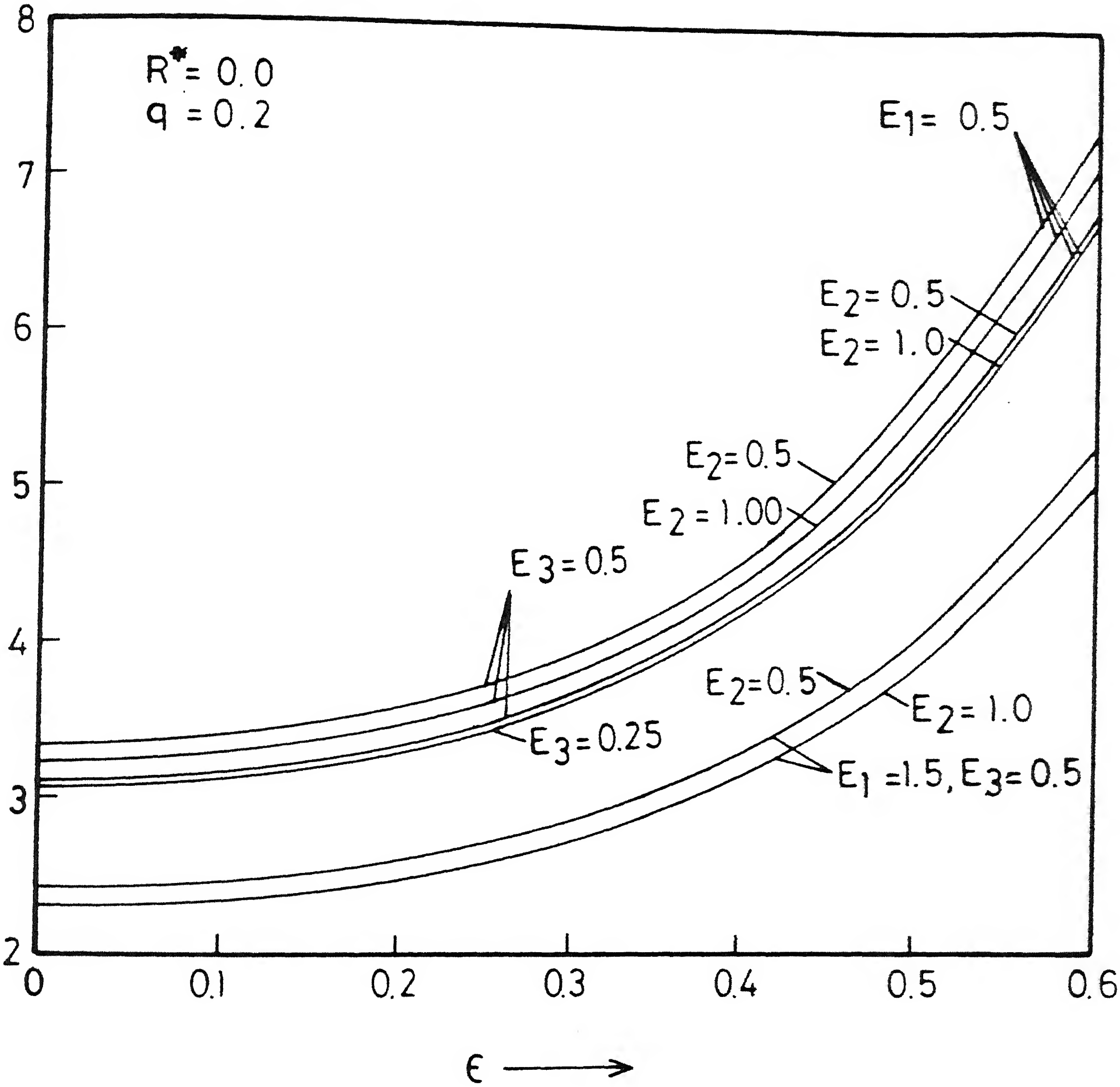


FIG. 4.11 VARIATION OF $(-\Delta P)$ WITH ϵ FOR DIFFERENT E_1, E_2 & E_3.

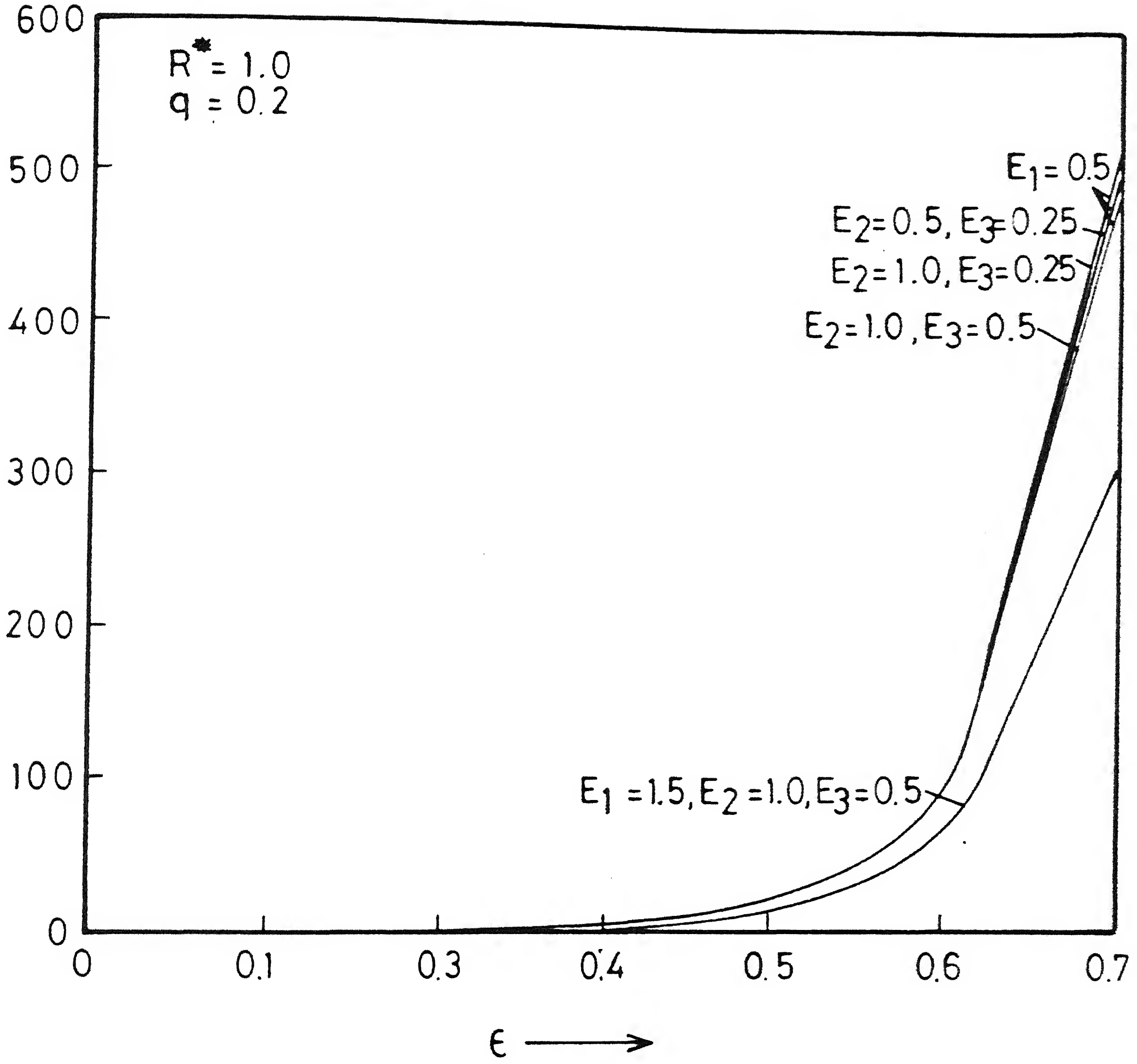
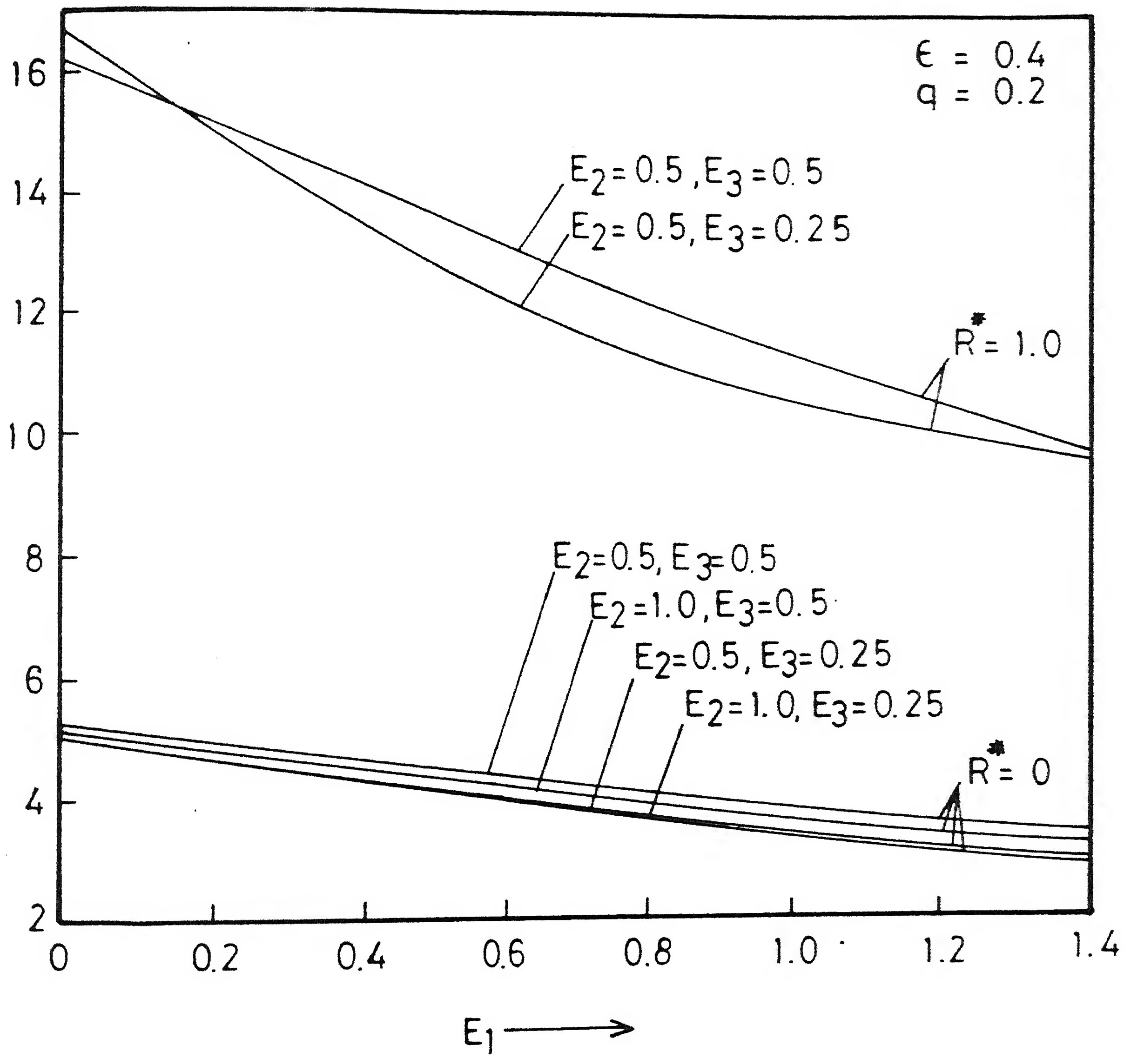


FIG.4.12 VARIATION OF $(-\Delta P)$ WITH ϵ FOR DIFFERENT E_1, E_2 & E_3.



4.13 VARIATION OF $(-\Delta P)$ WITH E_1 FOR DIFFERENT E_2, E_3 & R^*.

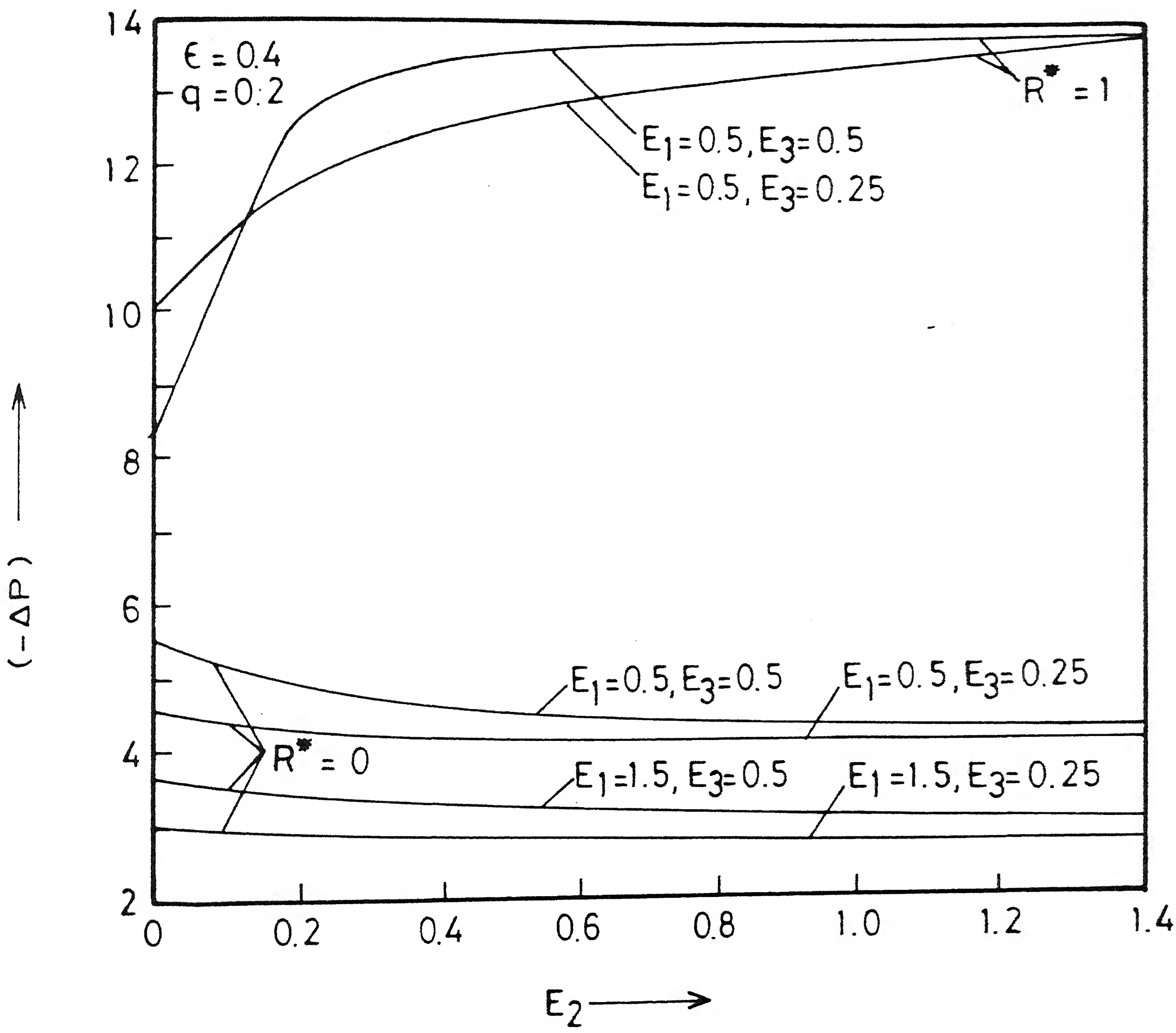


FIG.4.14 VARIATION OF $(-\Delta P)$ WITH E_2 FOR DIFFERENT E_1, E_3 & R^*.

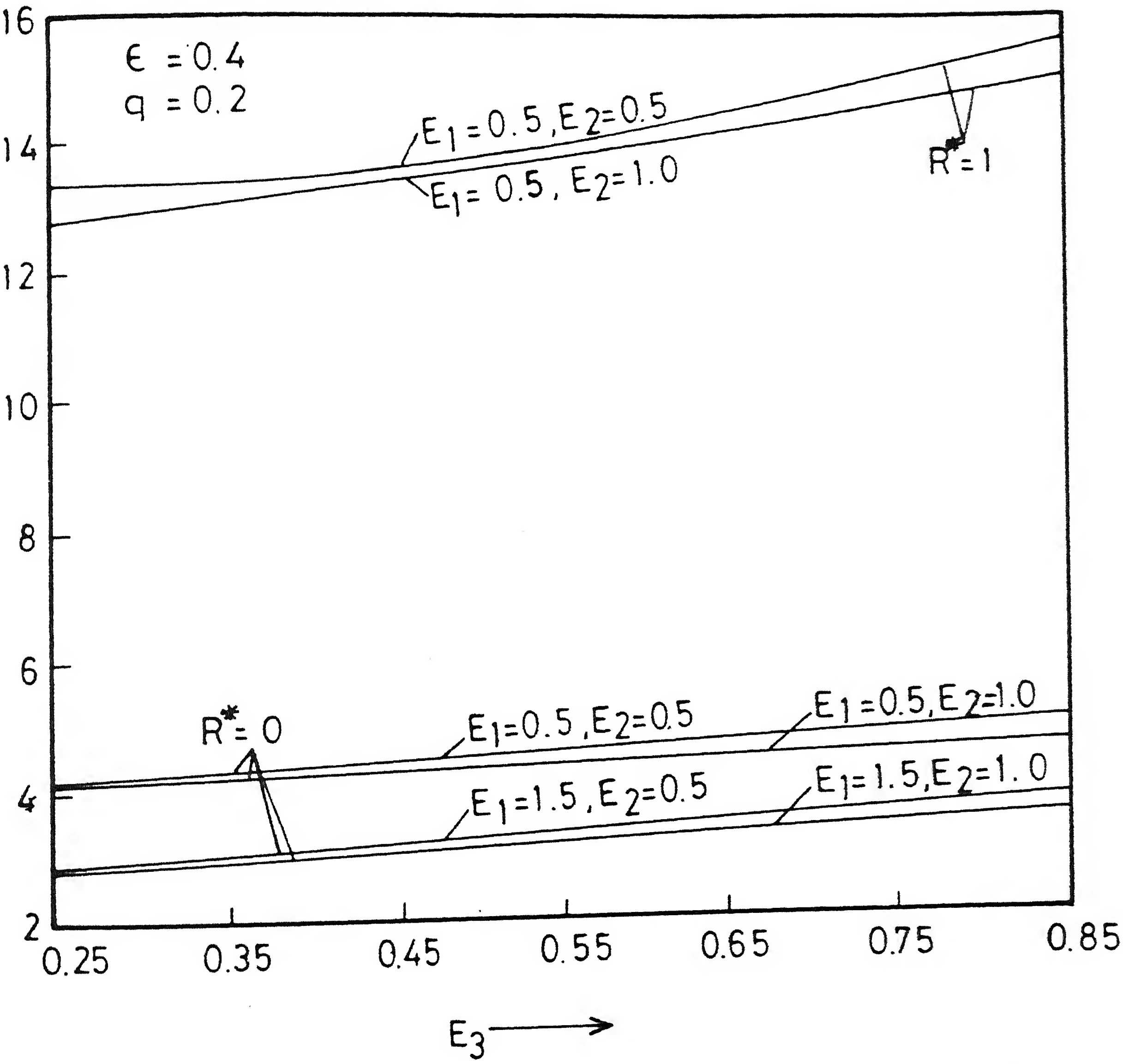


FIG. 4.15 VARIATION OF $(-\Delta P)$ WITH E_3 FOR DIFFERENT E_1, E_2 & R^*.

variation with the simple microfluid parameter becomes very small at $R^*=1.0$. These effects are further elaborated in Figs.(4.3 to 4.5) which are drawn for $\varepsilon=0.4$. The variation of Δp with E_1 is more, compared to its variation with respect to E_2 and E_3. The effect of the parameter E_3 at $R^*=1.0$, depends upon the values of E_1 and E_2 e.g. pressure rise gets reduced with the increase in E_3 for $E_2 < 0.2$.

In Figs. (4.16 to 4.20) we have drawn shear stress at the wall across a wavelength (for $\varepsilon=0.1$). The shear stress profile changes singificantly with the variation in E_2. We also note that the shear stress at the wall increases as the amplitude ratio and the slope parameter increase. The variation is more significant for $R^*=5$; it is observed that $\bar{\tau}_w$ increases with increase in parameter E_1 at $R^*=0$ & 5, however the effect of the simple microfluid parameter E_2 on $\bar{\tau}_w$ is affected by the Reynolds number. At $R^*=0$, $\bar{\tau}_w$ shows a marginal decrease with an increase in E_2. But it shows significant increase with an increase in E_2 at $R^*=5$. Thus the Reynolds number plays an important role here.

4.4 CONCLUSION

It is noted that the pressure rise, friction force and the stress at the wall increase with increase in the amplitude ratio of the wave. Also they decrease in magnitude while the simple microfluid parameters E_1 and E_2 increase but increase with increase in the rotation parameter E_3. It is also noted that the pressure rise and the friction force decrease as the flow rate is

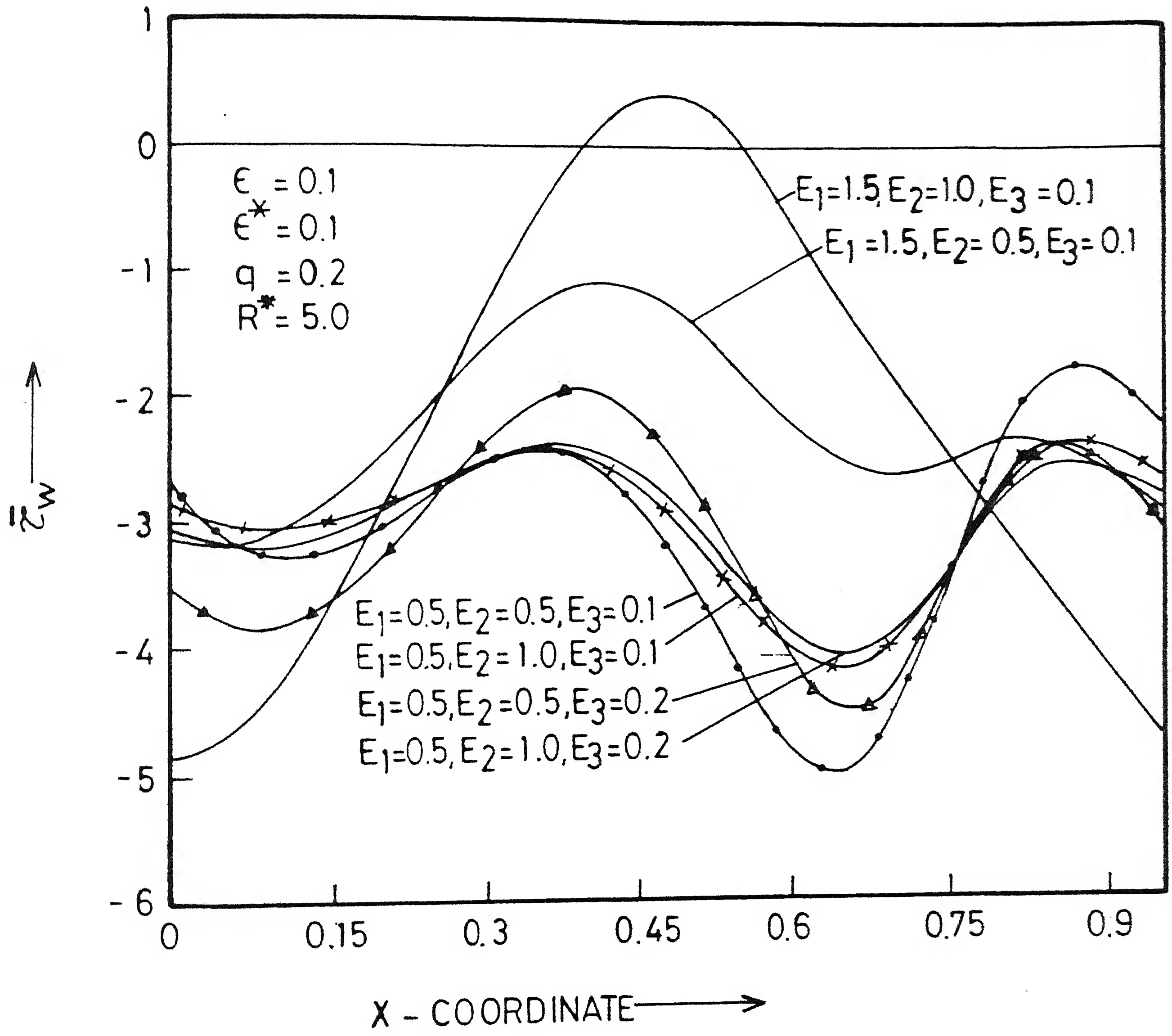


FIG. 4.16 VARIATION OF $\bar{\tau}_w$ ALONG THE AXIS FOR DIFFERENT E_1, E_2 & E_3.

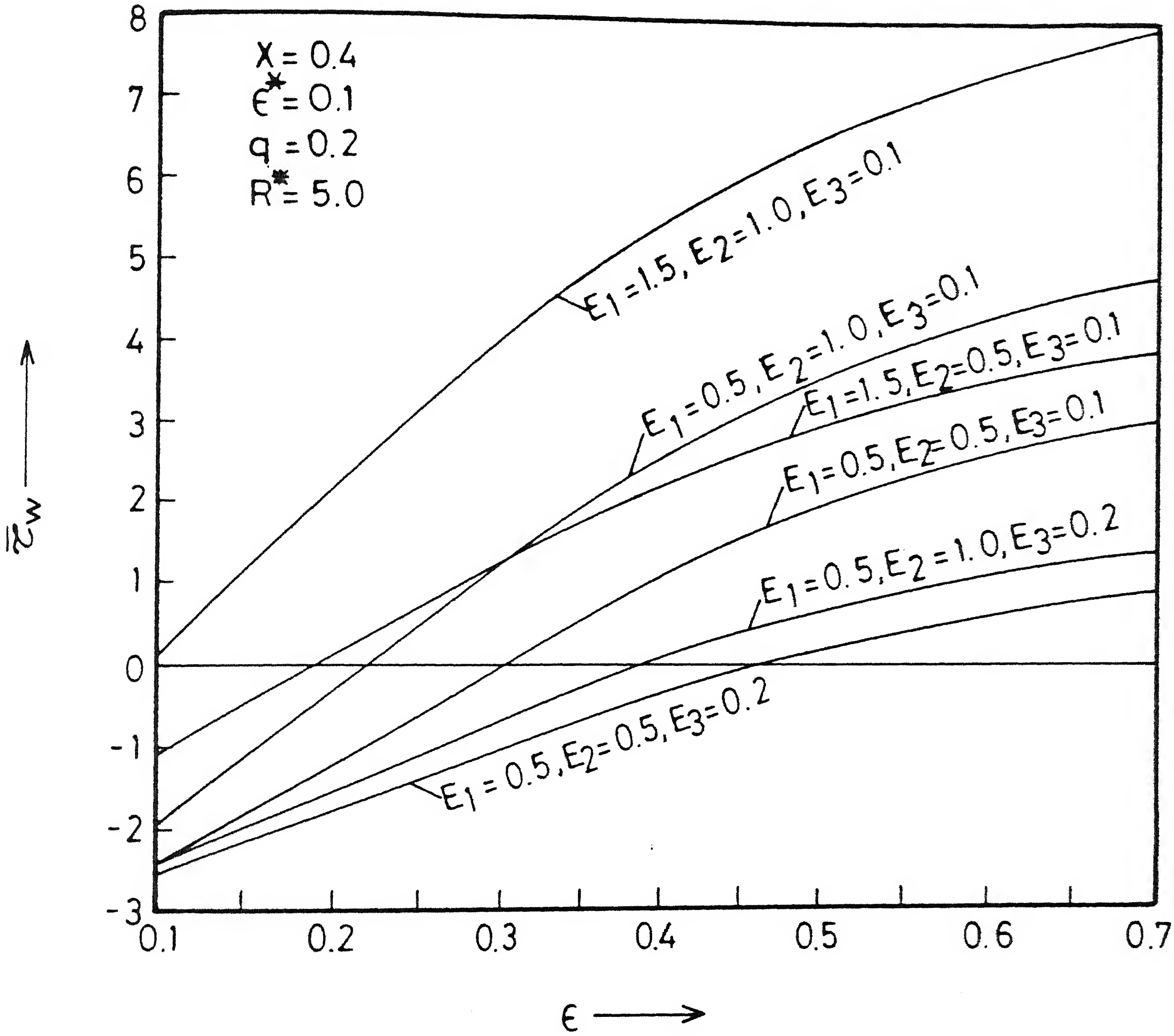


FIG.4.18 VARIATION OF $\bar{\tau}_w$ WITH ϵ FOR DIFFERENT E_1, E_2 & E_3.

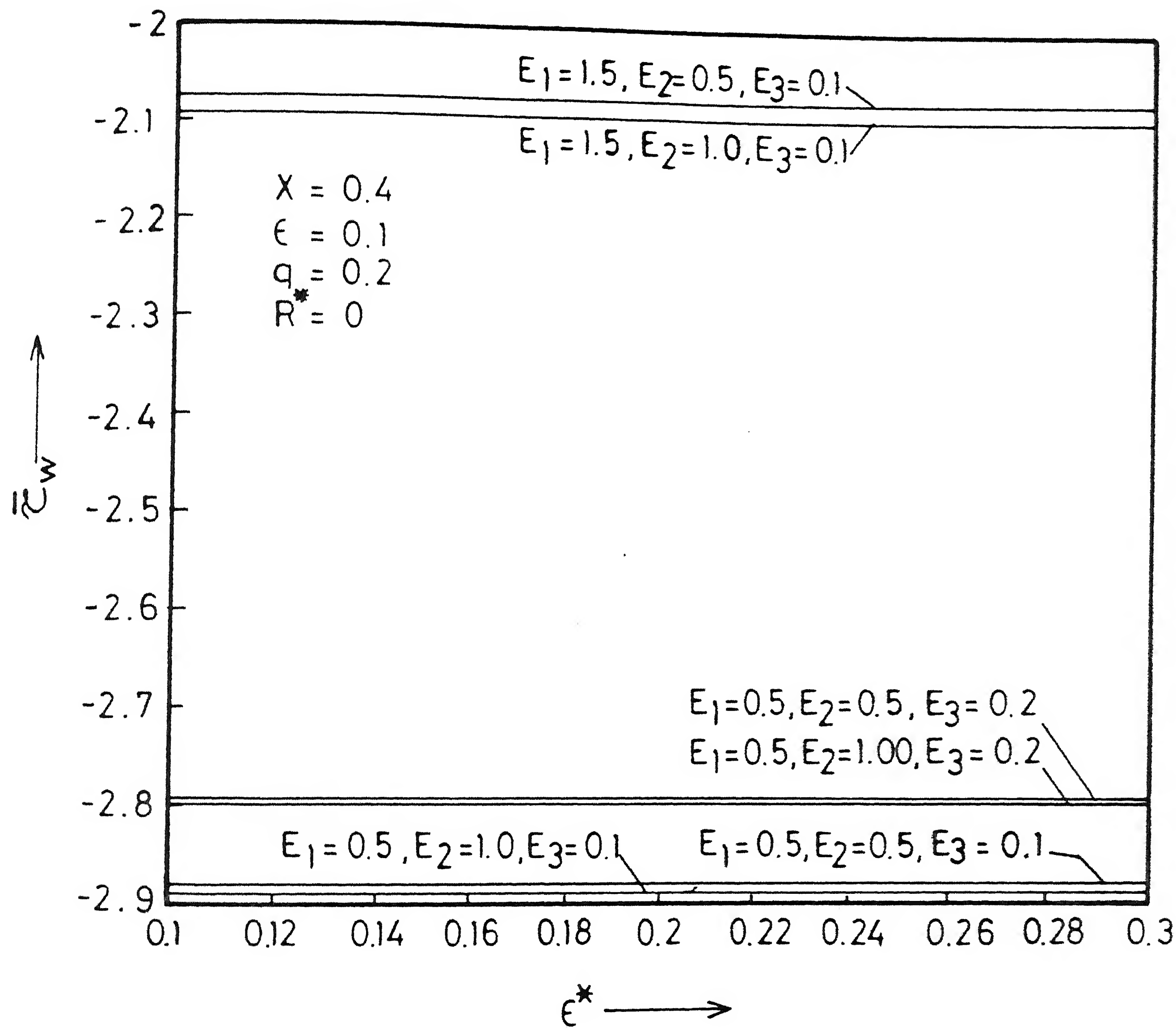


FIG.4.19 VARIATION OF $\bar{\tau}_w$ WITH ϵ^* FOR DIFFERENT E_1, E_2 & E_3.

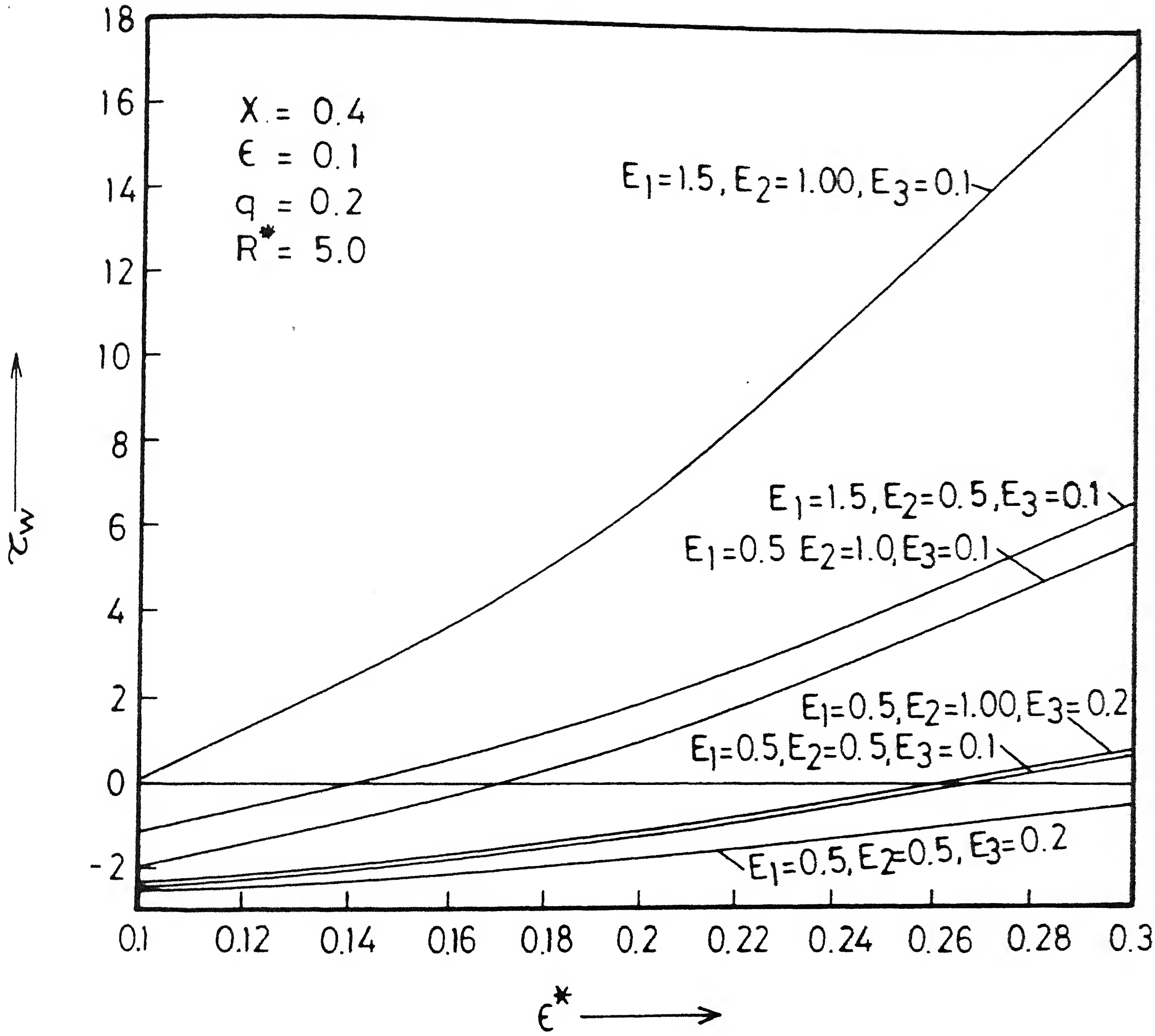


FIG. 4.20 VARIATION OF $\bar{\tau}_w$ WITH ϵ^* FOR DIFFERENT E_1, E_2 & E_3 .

increased. At $R^*=0$, $\bar{\tau}_w$ shows only a marginal decrease with an increase in E_2 but it significantly increases with increase in E_2 at $R^*=5$. Thus we can conclude that the Reynolds number plays an important role in the peristaltic motion of a simple microfluid.

CHAPTER V

SELF PROPULSION OF SPERMATOZOA IN MICROCONTINUA : EFFECT OF TRANSVERSE WAVE MOTION OF CHANNEL WALLS

5.1 INTRODUCTION

The spermatozoa are tiny living cells having their own motility. They move towards the oviduct through the mucus filling the cervical canal by sending waves of lateral displacement down a thin tail or flagellum. In recent years many research workers have analytically investigated swimming of micro organism in liquids. Taylor (1951) in his pioneering work on locomotion of an organism, possessing a single flagellum (e.g. spermatozoa), modelled the micro organim as a two dimensional infinite extensible sheet of zero thickness. He thus considered the motion of fluid near the sheet down which the waves of lateral displacements are propagated. Hancock (1953) studied the propulsion of a thin circular filament in infinite viscous fluid to obtain the velocity of propulsion of spermatozoa. Reynolds (1965) considered a two dimensional model and used Taylor's sheet approach to consider the effect of rigid walls adjacent to it. Brokaw (1972), Shack & Lardner (1974), Shack et al. (1974), Katz (1974), Lighthill (1976), Shack et al. (1978, 1988) and others have also attempted to study the motion of spermatozoa in cervical canal.

In all these works the problem has been treated from continuum point of view. However, there exists a strong likelihood that the cervical mucus be considered as a micro continuum fluid. Odebald (1959, 1962) has observed that the cervical mucus is a suspension of high molecular weight macro molecules in a mesh, having a spacing of 0.03 μm in them, Elstein (1971), Davajan et al. (1971). This motivated Sinha et al. (1982) to consider the micro continuum approach for studying self propulsion of spermatozoa in a micropolar fluid filled in a rigid channel. In a recent study, Shukla et al. (1988) have modelled the cervical canal as a channel with distensible walls. They have considered transverse waves of finite amplitude travelling down the flexible walls to account for the mascular activity.

In view of this we discuss here the self propulsion of spermatozoa through a microcontinuum filling the channel, with flexible walls. This chapter consists of two parts, in part I mucus has been modelled as micropolar fluid model while in part II, simple microfluid model in considered.

5.2 FORMULATION

Consider the motion of spermatozoa in cervical canal whose physical configuration is shown in fig. (5.1). The cervical canal is modelled as a two dimensional channel with flexible walls. It is assumed that the sinusoidal waves of finite amplitude are imposed along the flexible walls of the channel in

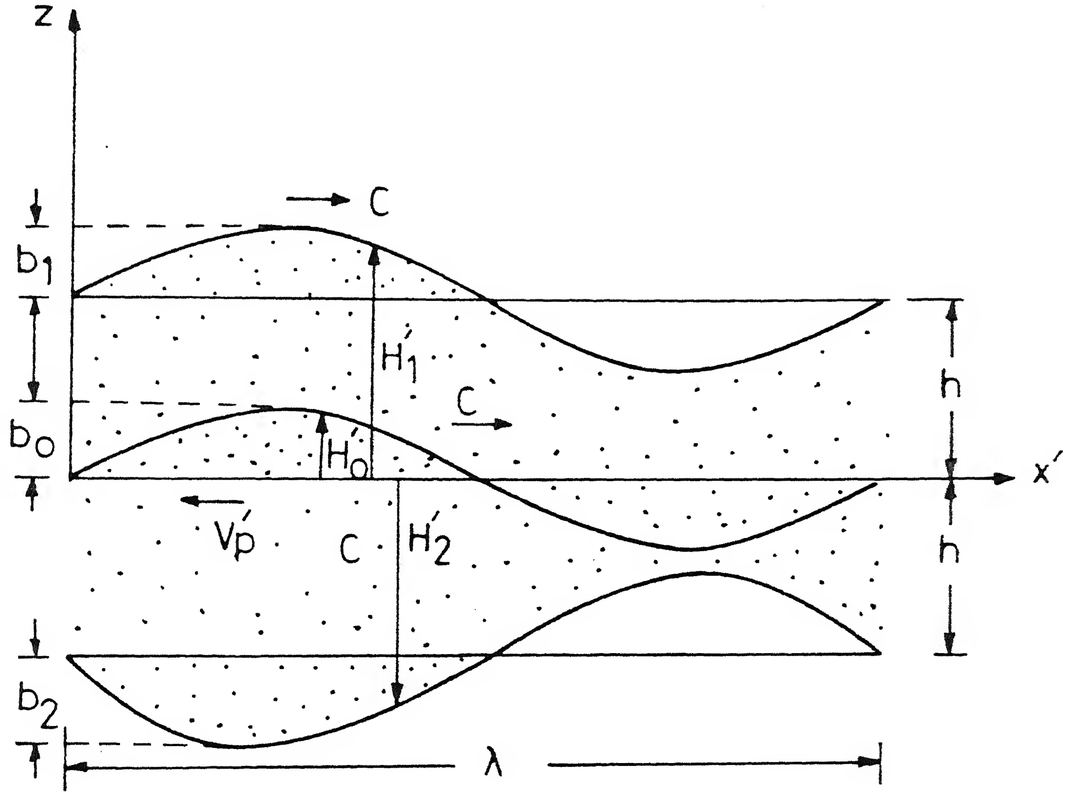


FIG.5·1 PROPAGATION OF AN ELASTIC SHEET SWIMMING THROUGH A TWO DIMENSIONAL CHANNEL WITH PERISTALTIC MOTION OF WALLS

the direction opposite to the motion of spermatozoa. The spermatozoa is assumed to be an infinite flexible sheet which propels itself forward by passing sinusoidal lateral waves down its length in the backward direction. The sheet is considered to be swimming with the propulsive velocity V_p' in the negative axial direction. Further we assume that the waves travelling along the channel walls and that along the sheet are in synchronization under the steady state (Shukla et al. 1988). Thus they have same wave speed C (along positive axial direction) and wavelength λ. In a fixed frame of reference (X', Z', t') the forms of the channel boundaries and the sheet at an instance t' are given by

$$Z' = h + b_1 \sin \frac{2\pi}{\lambda} (X' - Ct' + V_p' t') = H_1' (X', t') \tag{5.1a}$$

$$Z' = -h + b_2 \sin \frac{2\pi}{\lambda} (X' - Ct' + V_p' t') = H_2(X', t') \tag{5.1b}$$

$$Z' = b_0 \sin \frac{2\pi}{\lambda} (X' - Ct' + V_p' t') = H_0' (X', t') \tag{5.1c}$$

where b_1, b_2 are the amplitudes of the sinusoidal waves travelling along the upper wall and the lower wall respectively, b_0 is the amplitude of the lateral wave along the sheet and 2h is the width of the channel.

It may be noted that the relative velocity of the waves with respect to the fixed frame is $C-V_p'$. Hence in a moving frame (x', z', t') where $[x' = X' - Ct' - V_p t', z' = Z']$, moving with the velocity $C-V_p'$ in the positive axial direction, the sheet and

the walls appear to be stationary and the flow is steady. Thus in this moving frame of reference, the channel walls are given by $z' = h.h_1(x)$, $z' = h.h_2(x)$ and the sheet by $z' = h.h_0(x)$, where

$$h_1(x) = 1 + \varepsilon_1 \sin 2\pi x \tag{5.2a}$$

$$h_2(x) = -1 + \varepsilon_2 \sin 2\pi x \tag{5.2b}$$

$$h_0(x) = \varepsilon_o \sin 2\pi x \tag{5.2c}$$

$$\varepsilon_i = \frac{b_i}{h}, \quad i = 0,1,2, \qquad x = x'/\lambda.$$

Further the Reynolds number of motion based on the length of the sperm, density and viscosity of the surrounding fluid and velocity of propulsion is found to be of order 10^{-3} in such studies. Therefore we consider here that the motion is predominately due to viscous forces and the inertial forces have been neglected. In the following the superscript (+) refers to various quantities in the region $H_0' \le Z' \le H_1'$ while (-) refers to those in the region $H_2' \le Z' \le H_0'$.

5.3 PART I : FLOW THROUGH MICROPOLAR FLUID

5.3.1 Governing Equations

Here we consider mucus as a micropolar fluid. The equations governing the inertial free flow under the very long wavelength approximation have been deduced by Sinha et al. (1983) and are smimilar to those in lubrication theory for micropolar fluid [Prakash and Sinha 1975]. We reproduce these equations here in

the non-dimensional form with respect to the moving frame of reference. Thus for the region $h_o(x) \le z \le h_1(x)$ we write the equations as

$$-\frac{\partial p^+}{\partial x} + (1 + \frac{\bar{k}}{2}) \frac{\partial^2 u^+}{\partial z^2} + \bar{k}\frac{\partial \phi^+}{\partial z} = 0 \tag{5.3}$$

$$\frac{\partial p^+}{\partial z} = 0 \tag{5.4}$$

$$\bar{\nu}\frac{\partial^2 \phi^+}{\partial^2} - \bar{k}\frac{\partial u^+}{\partial z} - 2\bar{k}\,\phi^+ = 0 \tag{5.5}$$

where the following non-dimensional scheme has been used.

$$x = x'/\lambda \;;\quad z = z'/h \;;\quad t = Ct'/\lambda \;;\quad V_p = V_p'/C$$

$$u^+ = u'^+/C \;;\quad \phi'^+ h/C; \quad p^+ = \frac{p'^+ h^2}{\lambda C \mu} \tag{5.6}$$

$$\bar{\nu} = \frac{\nu}{\mu h^2} \;;\quad \bar{k} = \chi/\mu$$

[Here the p'^+ is the pressure, u'^+ the axial velocity, ϕ'^+ the microrotation vector χ, ν are viscosity coefficients for micropolar fluid].

It may be noted that $\bar{k}$ is coupling parameter as the equations become uncoupled as $\bar{k}\to 0$ and $\bar{\nu}$ is material coefficient parameter. The equations governing the flow in the region $h_2(x) \le z \le h_0(x)$ can also be written similarly.

The boundary conditions with respect to moving frame of reference are

$$u^{+} = u^{-} = -1 \,,\; \phi^{+} = \phi^{-} = 0 \quad \text{at} \quad z = h_0 \tag{5.7}$$

$$u^{+} = V_p - 1 \,,\; \phi^{+} = 0 \quad \text{at} \quad z = h_1 \tag{5.8}$$

$$u^{-} = V_p - 1 \,,\; \phi^{-} = 0 \quad \text{at} \quad z = h_2 \tag{5.9}$$

The boundary conditions for ϕ^{+}, ϕ^{-} denote that the fluid - solid interface interaction is so strong that the microstructure (fluid particle) does not rotate relative to the wall.

Further the sheet is self-propelling and the forces exerted by the fluid on it must balance for its motion to be with a constant velocity i.e.

$$\int_S (T^{+} + T^{-})\, dS = 0 \tag{5.10}$$

where T^{+} and T^{-} are the resultant of forces acting on the upper and lower surfaces of the sheet respectively and S is the surface area of the micro-organism [Shack & Lardener (1974), Shukla et al. (1988)].

5.3.2 Analysis

Solving the equations (5.3) and (5.5) using the boundary conditions for ϕ^{+} we get

$$\phi^{+} = P^{+} \left[\frac{h_1 \cosh \alpha(z-h_0) - h_0 \cosh \alpha(z-h_1)}{\sinh \alpha H_1} - \frac{1}{2} z \right.$$

$$+ \frac{1}{4}(h_1 + h_0) - (\frac{h_1 + h_0}{2})\left\{\frac{\cosh \alpha(z-h_0) - \cosh \alpha(z-h_1)}{H_1}\right\}\Bigg]$$

(5.11)

$$+ V_p \left[\frac{\text{cohh } \alpha(z-h_0) - \cosh \alpha(z - h_1)}{F_1(h_1) \sinh \alpha H_1} - \frac{1}{2}\right]$$

Substituting for ϕ^+ from eqn. (5.11) in eqn. (5.4) we get

$$u^+ = P^+\Bigg[\frac{1}{2}(z^2 - zh_0 - zh_1 + h_1h_0) - \frac{H_1 \alpha}{2L^2 \sinh \alpha H_1}\Big\{\cosh \alpha(z-h_0)$$

$$+ \cosh \alpha(z-h_1) - \cosh \alpha H_1 - 1\Big\} + V_p \frac{F_1(z)}{F_1(h_1)} - 1 \qquad (5.12)$$

$$\text{where } \alpha^2 = N^2L^2, \quad N = (\frac{\bar{k}}{2+\bar{k}})^{1/2}, \quad L = (\frac{4}{\bar{\nu}})^{1/2} \qquad (5.13)$$

$$F_1(z) = (z-h_0) - \frac{\alpha}{L^2 \sinh \alpha H_1}\left\{\cosh \alpha(z-h_0) - \cosh \alpha(z-h_1) + \cosh \alpha H_1 - 1\right\}$$

(5.14)

$H_1 = (h_1 - h_0)$ and $P^+ = \frac{\partial p^+}{\partial x}$

Proceeding in the similar way for the other region also, i.e. $(h_2(x) \leq z \leq h_o(x))$ we obtain,

$$u^- = P^- \left[\frac{1}{2} (z^2 - zh_0 - zh_2 + h_2h_0) - \frac{H_2\alpha}{2L^2 \sinh \alpha H_2} \left\{ \cosh \alpha(z-h_0) \right. \right.$$

$$\left. \left. + \cosh \alpha(z-h_2) - \cosh \alpha H_2 - 1 \right\} \right] V_p \frac{F_2(z)}{F_2(h_2)} - 1 \qquad (5.15)$$

$$F_2(z) = (z-h_0) - \frac{\alpha}{L^2 \sinh \alpha H_2} \left\{ \cosh \alpha(z-h_0) - \cosh \alpha(z-h_2) + \cosh \alpha H_2 - 1 \right\}$$

(5.16)

$$H_2 = (h_2 - h_0) \quad \text{and} \quad P^- = \frac{\partial p^-}{\partial x}$$

The dimensionless flow flux $q^{\pm}$ (= $q'^{\pm}$ in the moving frame of reference is obtained from :

$$q^+ = \int_{h_0}^{h_1} u^+ \, dz \quad \text{and} \quad q^- = \int_{h_2}^{h_0} u^- \, dz$$

which on using eqns. (5.12) and (5.15) gives

$$q^+ = - P^+ G_1 + V_p G_2 - H_1 \qquad (5.17)$$

$$q^- = P^- G_3 - V_p G_4 + H_2 \qquad (5.18)$$

where

$$G_1 = H_1^3/12 + \frac{\alpha H_1^2}{2L^2 \sinh \alpha_1 H} \left\{ \frac{2\sinh \alpha H_1}{\alpha H_1} - \cosh \alpha H_1 - 1 \right\}$$

$$G_2 = \frac{1}{F_1(h_1)}\left[\frac{H_1^2}{2} - \frac{\alpha H_1}{L^2 \sinh \alpha H_1}\left\{\cosh \alpha H_1 - 1\right\}\right]$$

$$G_3 = H_2^3/12 + \frac{\alpha H_2^2}{2L^2 \sinh \alpha H_2}\left\{\frac{2\sinh \alpha H_2}{\alpha H_2} - \cosh \alpha H_2 - 1\right\}$$

$$G_4 = \frac{1}{F_2(h_2)}\left[\frac{H_2^2}{2} - \frac{\alpha H_2}{L^2 \sinh \alpha H_2}\left\{\cosh \alpha H_2 - 1\right\}\right]$$

Eqns. (5.17) and (5.18) can be rewritten and the pressure gradients $\partial p^{\pm}/\partial x$ can be obtained as,

$$\frac{\partial p^+}{\partial x} = -\frac{q^+}{G_2} + V_p\frac{G_2}{G_2} - \frac{H_1}{G_2} \tag{5.19}$$

$$\frac{\partial p^-}{\partial x} = \frac{q^-}{G_3} + V_p\frac{G_4}{G_3} - \frac{H_2}{G_3} \tag{5.20}$$

Integrating the equation of continuity it can be noted that the two fluxes, q^+ and q^- are constants. Further, it is evident that Δp, the pressure rise over a wavelength is same for the two regions. Therefore, integrating eqns. (5.19) and (5.20) over a wavelength we get the following relations among the three unknown quantities V_p, q^+ and q^-.

$$-q^+I_{11} + V_pI_{12} - I_{13} = \Delta p \tag{5.21}$$

$$q^{-} I_{21} + V_p I_{22} - I_{23} = \Delta p \tag{5.22}$$

where

$$I_{11} = \int_0^1 \frac{dx}{G_1} \; ; \qquad I_{21} = \int_0^1 \frac{dx}{G_3}$$

$$I_{12} = \int_0^1 \frac{G_2}{G_1}\, dx \; ; \qquad I_{22} = \int_0^1 \frac{G_4}{G_3}\, dx \; ;$$

$$I_{13} = \int_0^1 \frac{H_1}{G_1}\, dx \; ; \qquad I_{23} = \int_0^1 \frac{H_2}{G_3}\, dx \tag{5.23}$$

Using the stress-strain relationship and long wavelength approximation we write the force equilibrium condition, eqn. (5.10) as,

$$\int_0^1 \left[P \right] dx = 0 \tag{5.24}$$

$$\int_0^1 \left\{ \left(\frac{2+\bar{K}}{2}\right) \right\} \left[\frac{\partial u}{\partial z} \right]_{z=h_o(x)} + \frac{dh_o}{dx} \left[P \right] = 0 \tag{5.25}$$

which gives the third necessary relationship to determine V_p, q^{+} and q^{-}. Here [x] indicates the difference in the quantity x above and below the sheet.

Eqn. (5.25) on substituting for $\frac{\partial u^{\pm}}{\partial z}$, gives

$$q^{+}I_{31} + q^{-}I_{32} + V_p I_{33} + I_{34} = 0 \tag{5.26}$$

where

$$I_{31} = \int_0^1 \frac{1}{G_1}(h_o + H_1/2)\,dx \quad ; \quad I_{32} = \int_0^1 \frac{1}{G_3}(h_o + H_2/2)\,dx$$

$$I_{33} = \int_0 \left[(h_o + \frac{H_2}{2})\frac{G_4}{G_3} - (h_o + \frac{H_2}{2})\frac{G_2}{G_1} + \left\{\frac{1}{F_1(h_1)} - \frac{1}{F_2(h_2)}\right\}\right] dx$$

$$I_{34} = \int_0^1 \left[(h_o + \frac{H_1}{2})\frac{H_1}{G_1} - (h_o + \frac{H_2}{2})\frac{H_2}{G_3}\right] dx \tag{5.27}$$

Solving eqns. (5.21), (5.22) and (5.26) we get,

$$V_p = \frac{\left[\Delta p\left(\frac{I_{31}}{I_{11}} - \frac{I_{32}}{I_{21}}\right) + \frac{I_{31}I_{13}}{I_{11}} - \frac{I_{32}I_{23}}{I_{21}} - I_{34}\right]}{\left[I_{12}I_{31}/I_{11} - I_{32}I_{22}/I_{21} + I_{33}\right]} \tag{5.28}$$

The total flux $q = q^{+} + q^{-}$ is

$$q = \left[\frac{1}{I_{21}} - \frac{1}{I_{11}}\right]\Delta p + \left[\frac{I_{12}}{I_{11}} - \frac{I_{22}}{I_{21}}\right]V_p + \left[\frac{I_{23}}{I_{21}} - \frac{I_{13}}{I_{11}}\right] \tag{5.29}$$

Now, we can obtain the flux Q in the stationary frame and Q is related with flux, q in the moving frame by the following relation [Shukla et al. 1988],

$$Q = q + \left[(1 - V_p)\,(H_2 - H_1)/h\right] \tag{5.30}$$

Averaging this Q over a time period, we get the time averaged flow $\bar{Q}$ in terms of q and V_p as

$$\bar{Q} = q + 2\,(1 - V_p) \tag{5.31}$$

5.3.3. RESULTS AND DISCUSSION

In our study here the additional parameters, as compared to the study of shack and Lardner (1974), are the micropolar parameters N and L & the amplitude parameters ε_1 and ε_2. The parameters N and L arise as a consequence of rotational viscosity and the material coefficient. As the parameter $N \to 0$ i.e. χ (rotational viscosity) $\to 0$ the results reduce to those for the Newtonian fluid. The limiting case of $L \to \infty$ also gives Newtonian results. Thus, the present analysis in these limiting cases, yields the results of Shukla et al. (1988), for the particular case of no longitudinal motion of the walls. The parameters ε_1 and ε_2 represent the amplitudes of the waves on the channel walls. The case of $\varepsilon_2 = -\varepsilon_1$ is equivalent to imposing peristaltic motion on the walls. While $\varepsilon_1 = \varepsilon_2 = 0$ imply no wave motion on the wall. Thus, for no wave motion on the wall, and $\Delta p=0$ the present study reduces to that of Sinha et al. (1982).

We have numerically computed the values of propulsive velocity V_p and the time average flux $\bar{Q}$ for different values of

micropolar parameters and $\Delta p = 0.0, -0.05$, $\epsilon_1 = 0.0, 0.35$ and $\epsilon_2 = 0.0, -0.25$. The results are plotted in the figures (5.2 to 5.14). The propulsive velocity ,V_p, versus ϵ_o is plotted in figures (5.2 to 5.5), while its variation with respect to L is shown in figures (5.6 to 5.9). Figures (5.2 and 5.3) are plotted for $\epsilon_1 = 0 = \epsilon_2$ and Δp -0.05. They show that for a given micropolar fluid the propulsive velocity increases as magnitude of ϵ_o increases. This is similar to the behaviour as observed by Shack & Lardner (1974). Further, at a given amplitude ϵ_o, the propulsive velocity V_p shows, significant quantitative increase with the parameter N. In figures (5.4 and 5.5) we have shown the effect of wall amplitude variation for a given micropolar fluid. The propulsive velocity decreases as ϵ_1 increases, and for given ϵ_1 it further reduces as magnitude of ϵ_2 increases. Thus the effect of transverse wave of the walls is to reduce the speed of the spermatozoa. In fact for certain combinations of the wave amplitudes the direction of the propulsion can even be reversed. From figures (5.6 to 5.8) it is observed that for $\epsilon_o = 0.4$, V_p initially increases with L and then shows a decreasing trend, i.e. the propulsive velocity curve reaches maximum and then tends to the Newtonian value on $L \to \infty$. The value of L at which the maximum occurs very much depends upon the parameter N and the curve approaches maximum at smaller values of L as $N \to 1$. Further, it is observed from Figs. (5.5 and 5.6) that the effect of peristaltic wall motion is to reduce V_p considerably. In the case of no wave motion on the walls, V_p shows slight decrease as Δp decreases from 0.0 to -0.05, and this decrease becomes more

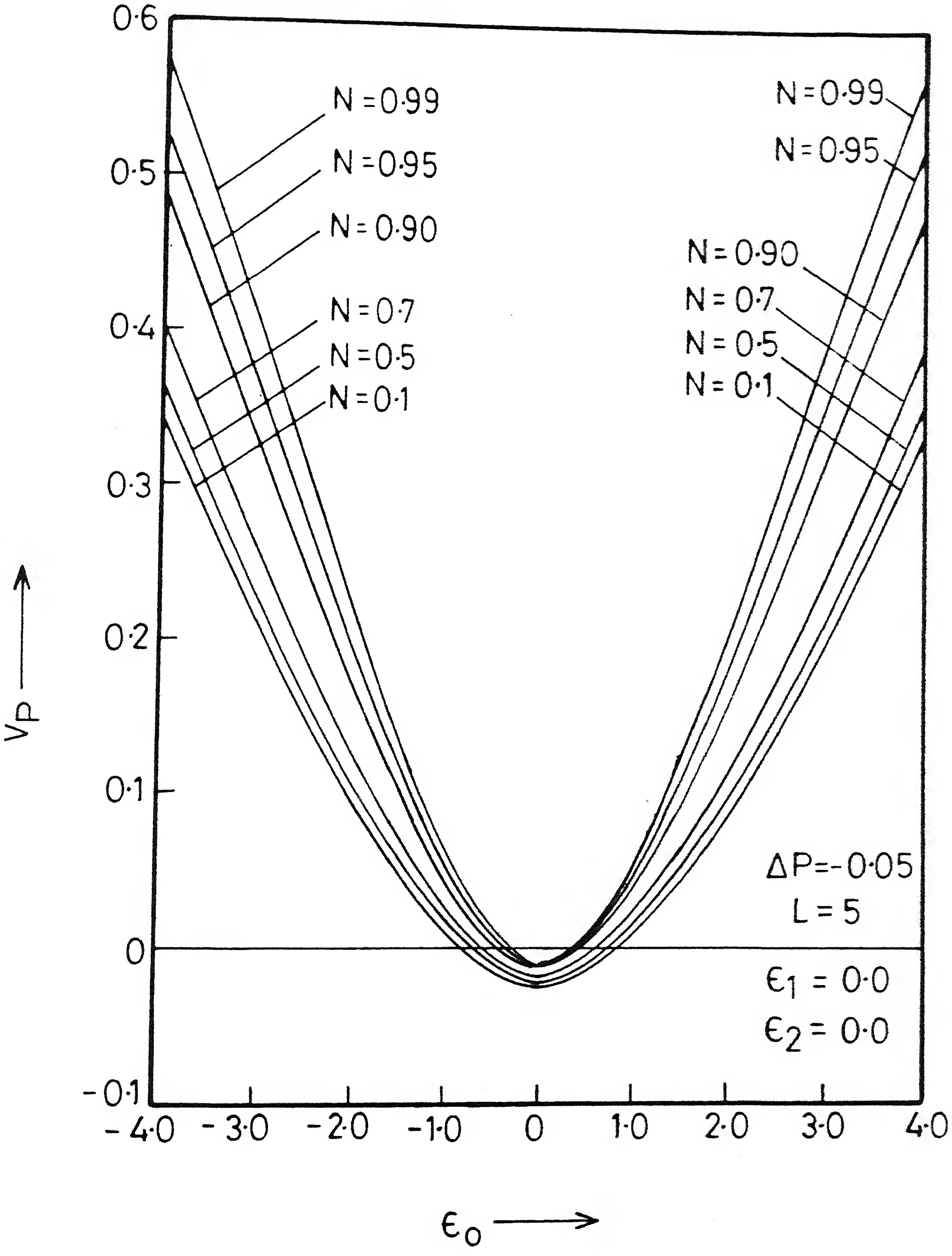


FIG.5·2 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT N

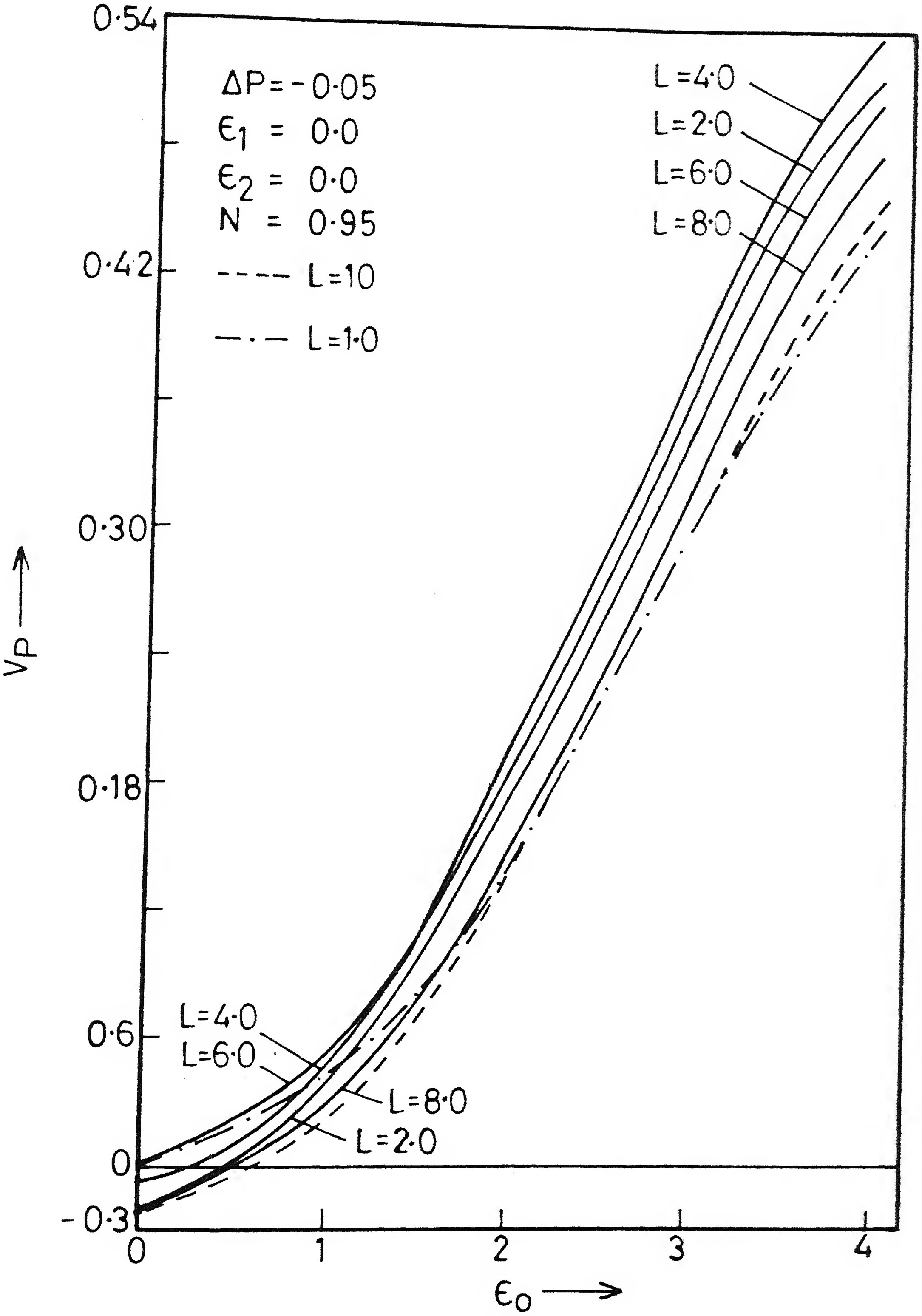


FIG.5·3 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT L

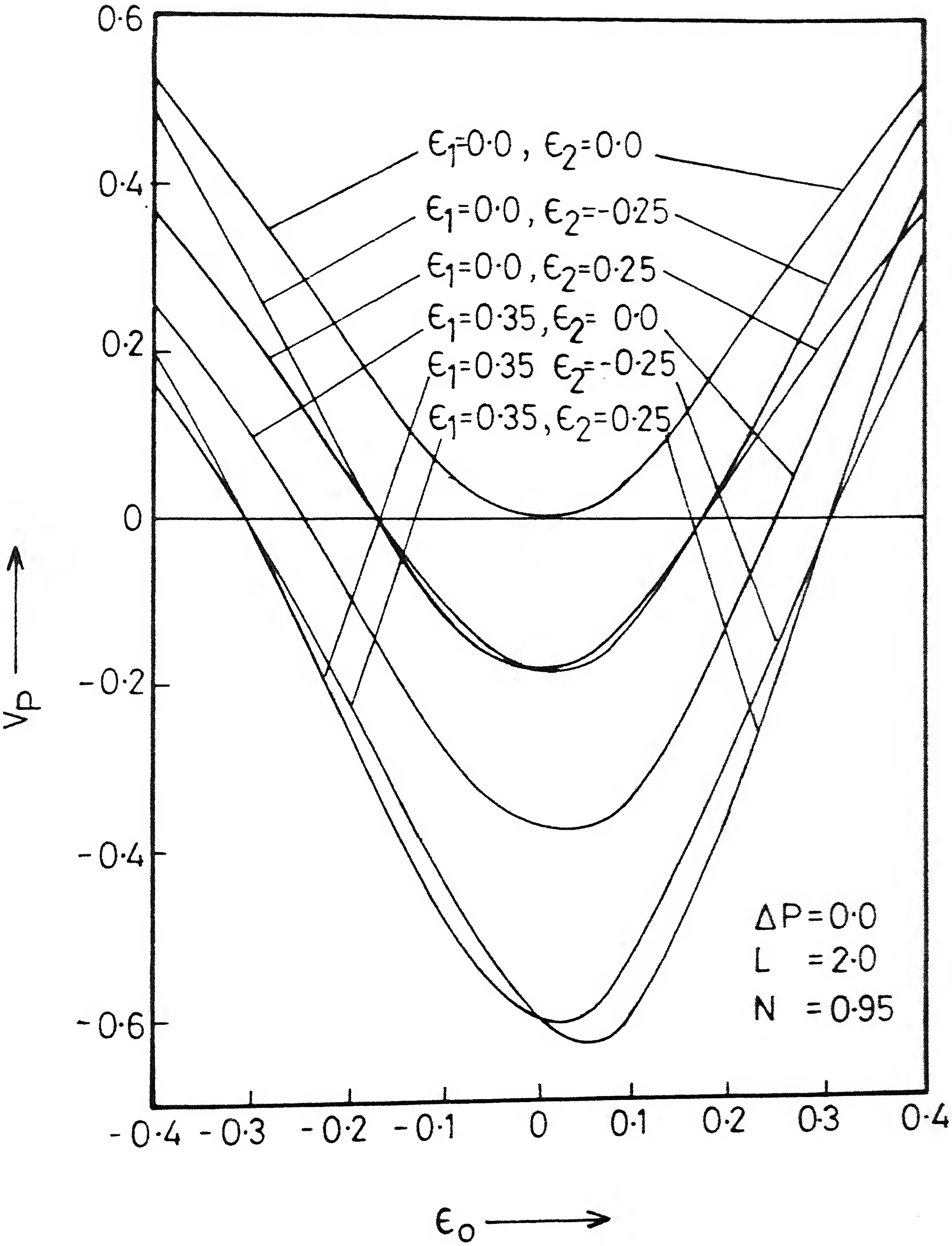


FIG. 5.4 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT ϵ_1 & ϵ_2

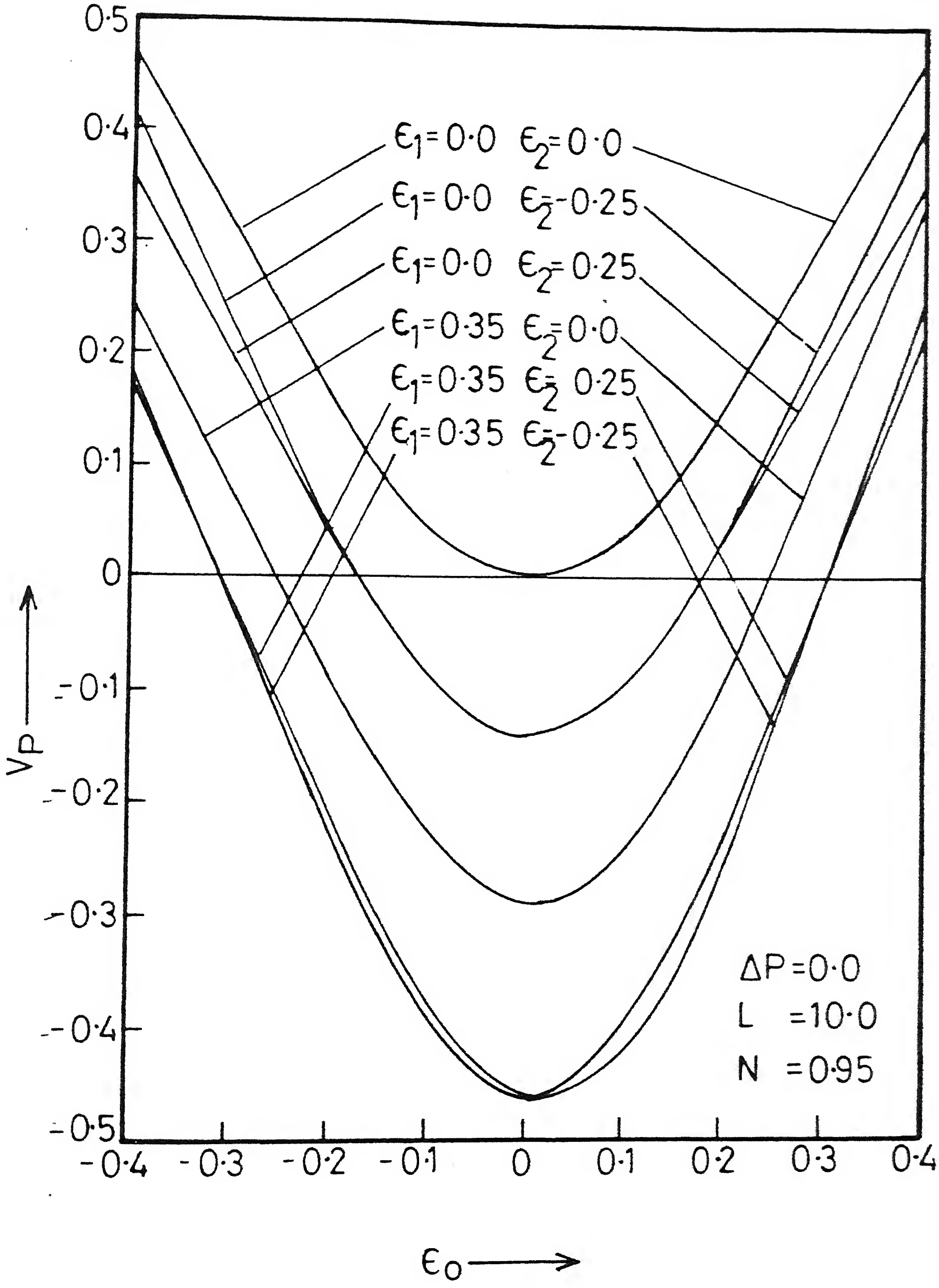


FIG. 5·5 VARIATION OF V_P WITH ϵ FOR DIFFERENT ϵ_1 & ϵ_2

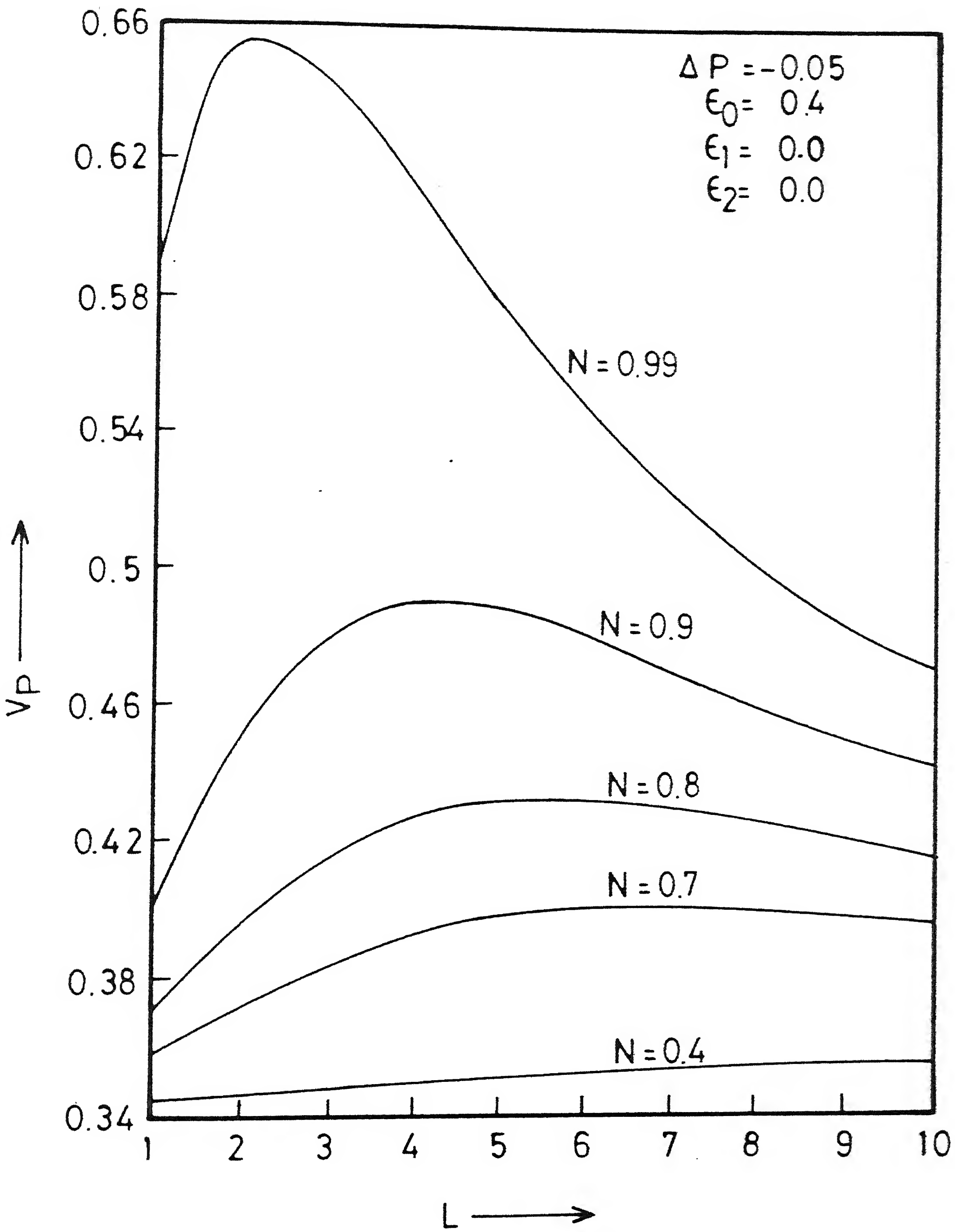


FIG.5.6 VARIATION OF V_P WITH L FOR DIFFERENT N.

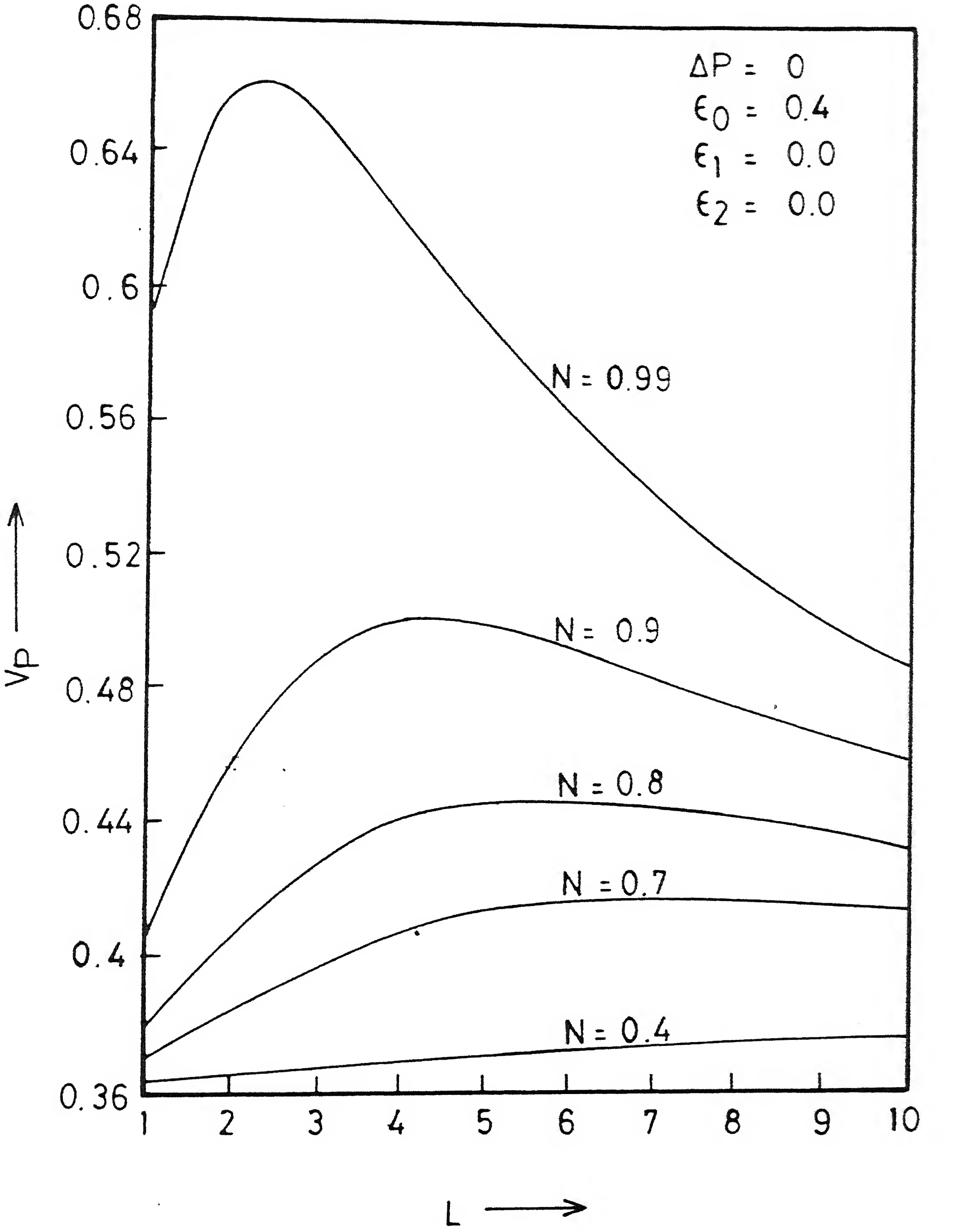


FIG.5.7 VARIATION OF V_P WITH L FOR DIFFERENT N.

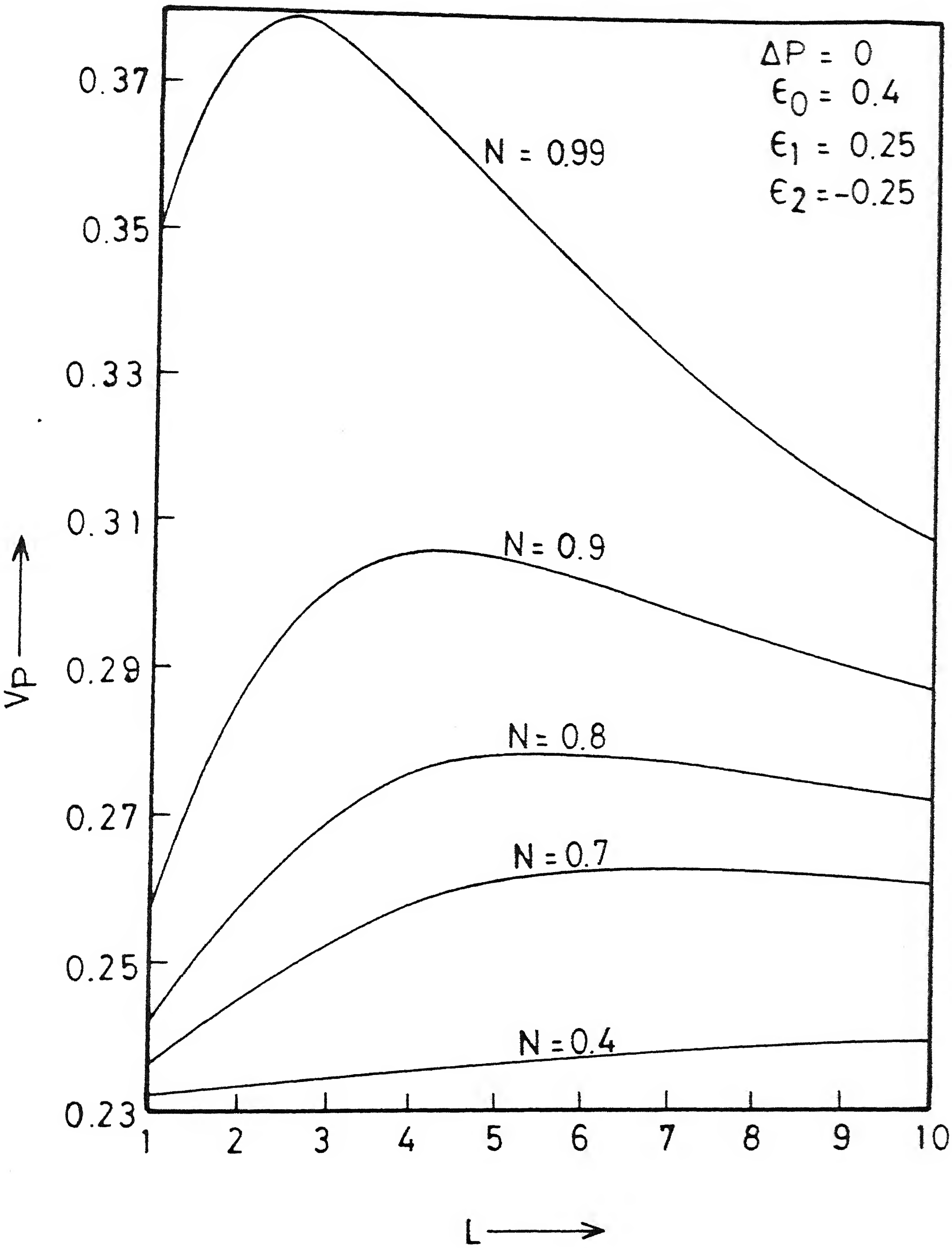


FIG.5.8 VARIATION OF V_P WITH L FOR DIFFERENT N.

pronounced as L increases or N decreases (Figs. 5.4 to 5.5). Fig. (5.9) shows the propulsion of a rigid sheet (ε_o=0) under the peristaltic waves on the channel walls. It shows the opposite behaviour as observed in the case of ε_o = 0.4.

In figures (5.10 to 5.14) time average flux ($\bar{Q}$) is plotted versus different parameters. The trend of $\bar{Q}$ with respect to various parameters is opposite to that of the propulsive velocity V_p. It is noted that $\bar{Q}$ decreases with the increase in the magnitude of ε_o and the effect of peristaltic wave on the wall is to increase $\bar{Q}$ considerably. The variation of $\bar{Q}$ with the micropolar parameters N and L depends upon the other parameters. For example, with peristaltic wave on the walls $\bar{Q}$ decreases with N at ε_o = 0.4 while it increases with N when ε_o = 0.0. It may be pointed out here that the results are in comfirmity with the results of Shack & Lardner (1974), Sinha et al. (1983) and Shukla et al. (1988).

5.4 PART II : FLOW THROUGH SIMPLE MICROFLUID

5.4.1 Governing equations

In this section we consider simple microfluid model for mucus. Thus, with the following non-dimensionalization scheme,

$$p = \frac{p' h^2}{\mu C \lambda} \quad ; \quad x = \frac{x'}{\lambda} \quad ; \quad z = \frac{z'}{h} \quad ; \quad t = \frac{Ct'}{\lambda}$$

$$u = \frac{u'}{C} \quad ; \quad \nu_{(13)} = \frac{h}{C} \nu'_{(13)} ; \quad \nu_{[31]} = \frac{h}{C} \nu'_{[31]}$$

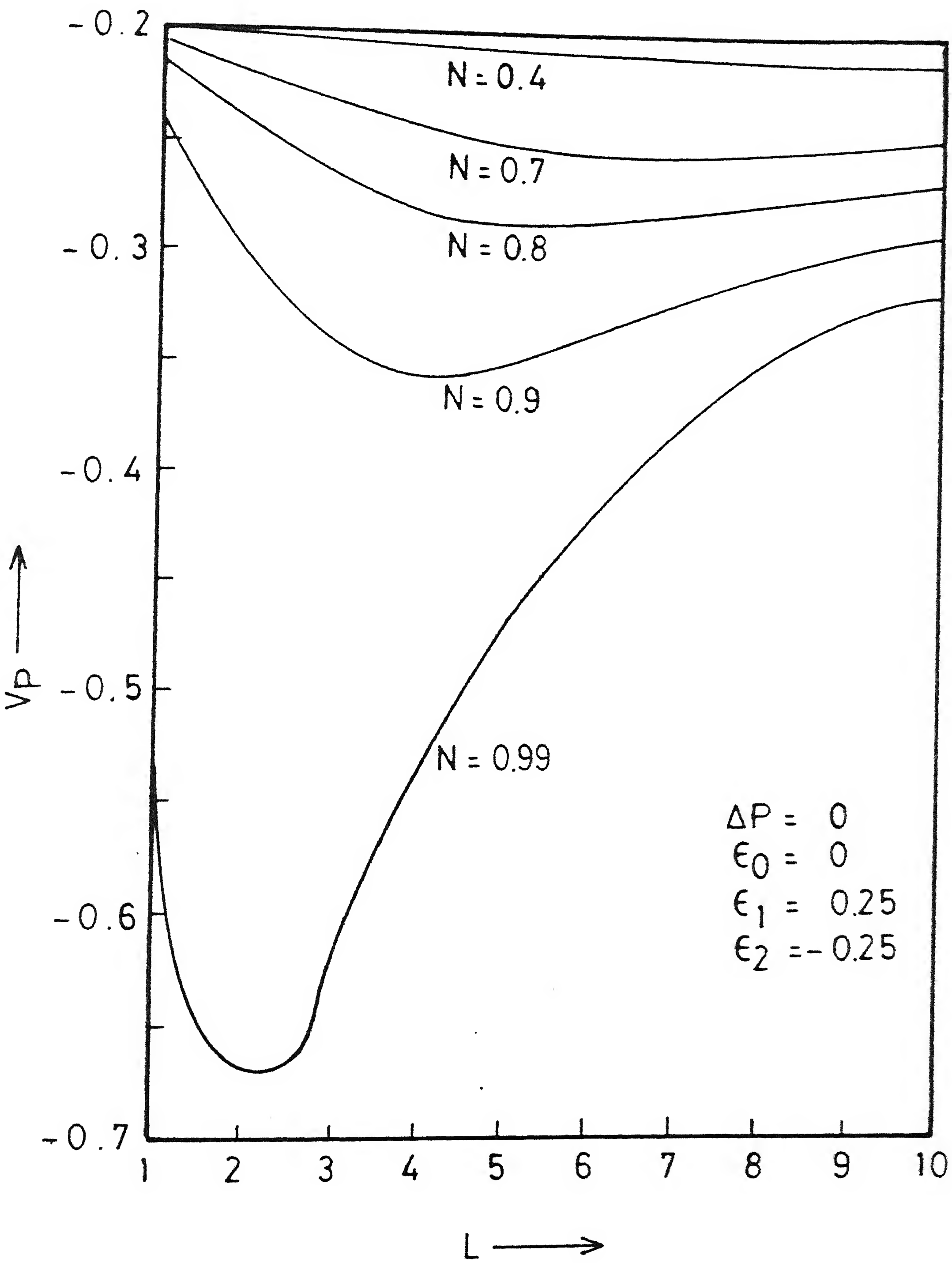
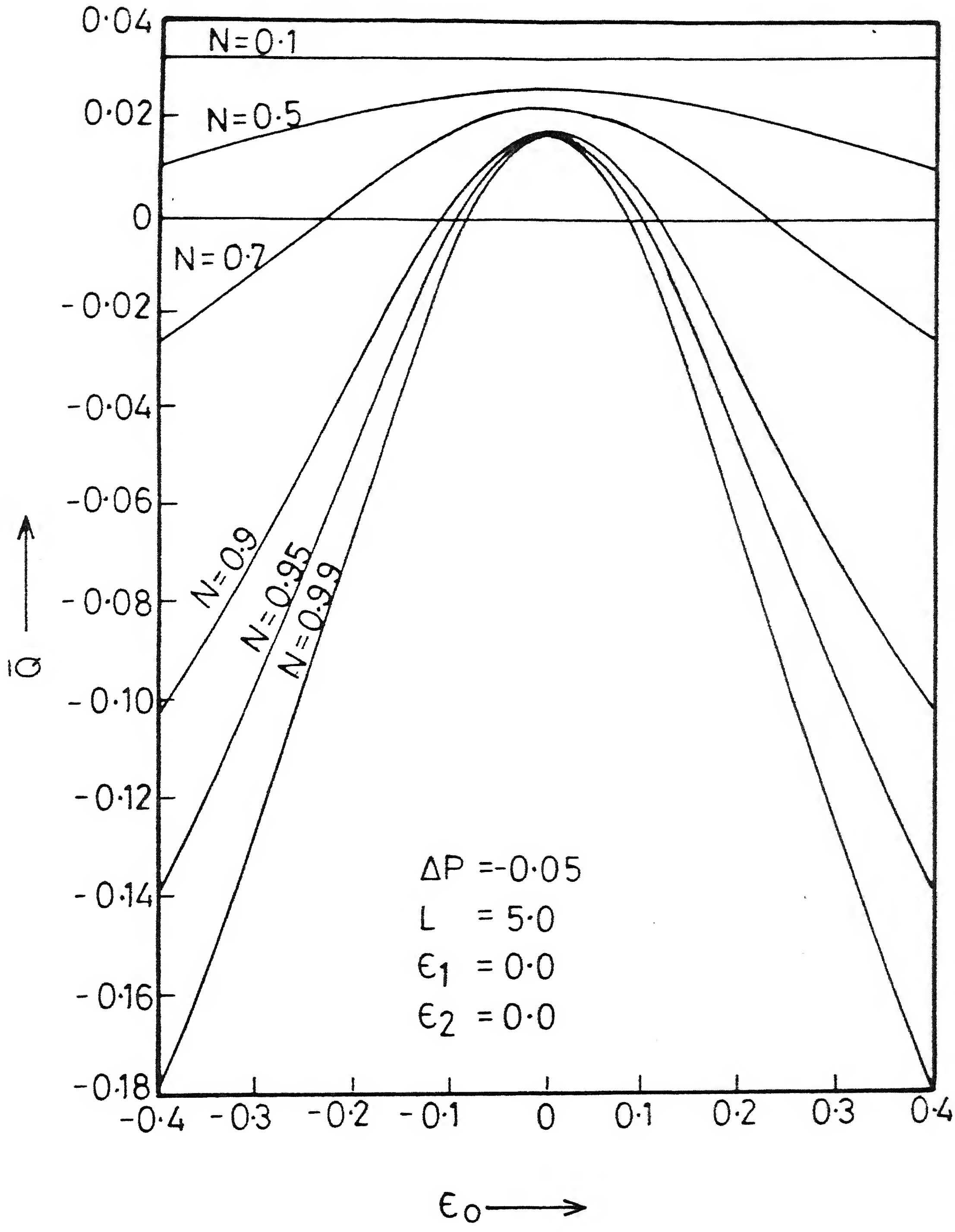


FIG.5.9 VARIATION OF V_P WITH L FOR DIFFERENT N.



FIG·5·10 VARIATION OF $\bar{Q}$ WITH ϵ_0 FOR DIFFERENT N

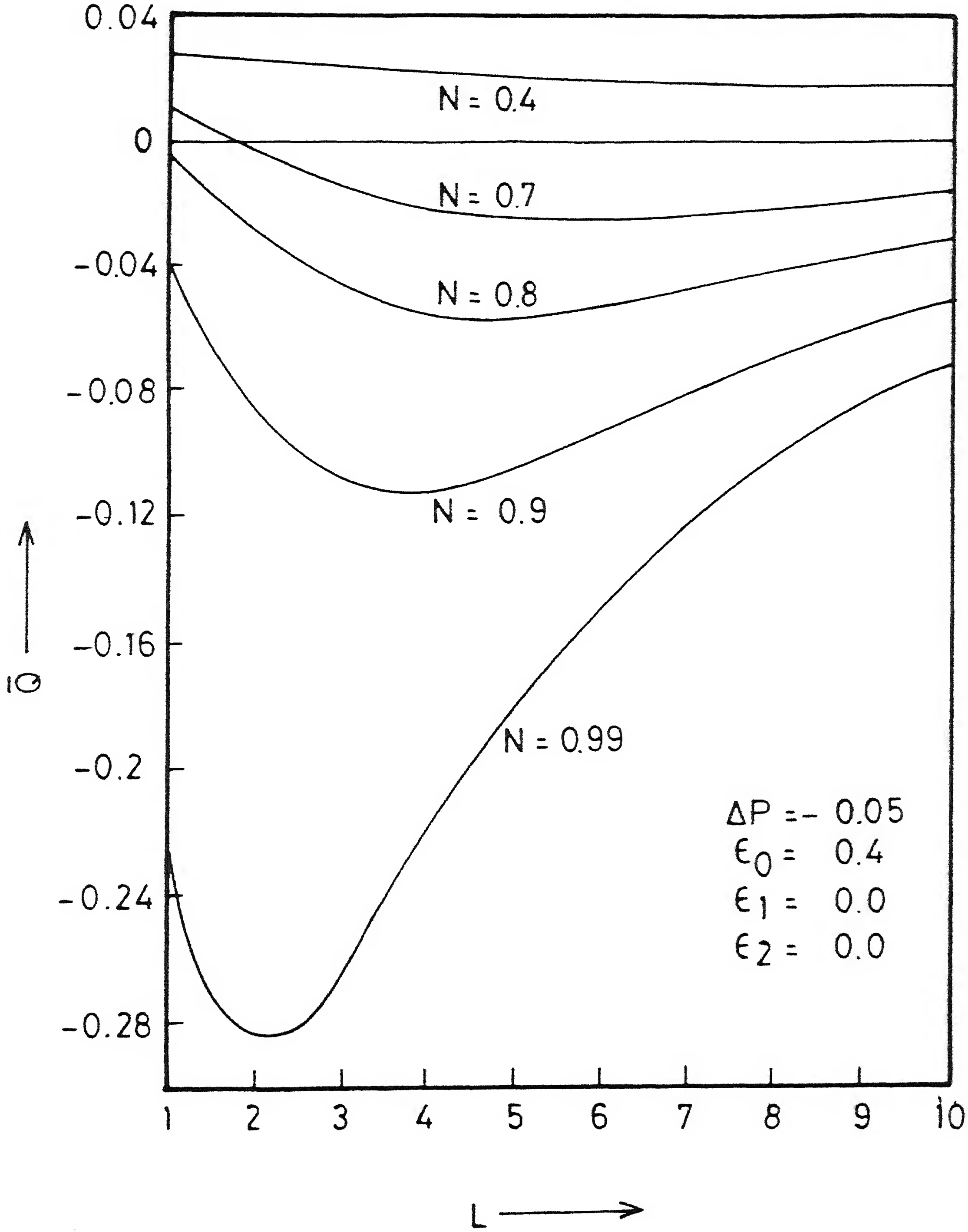


FIG.5.11 VARIATION OF $\bar{Q}$ WITH L FOR DIFFERENT N .

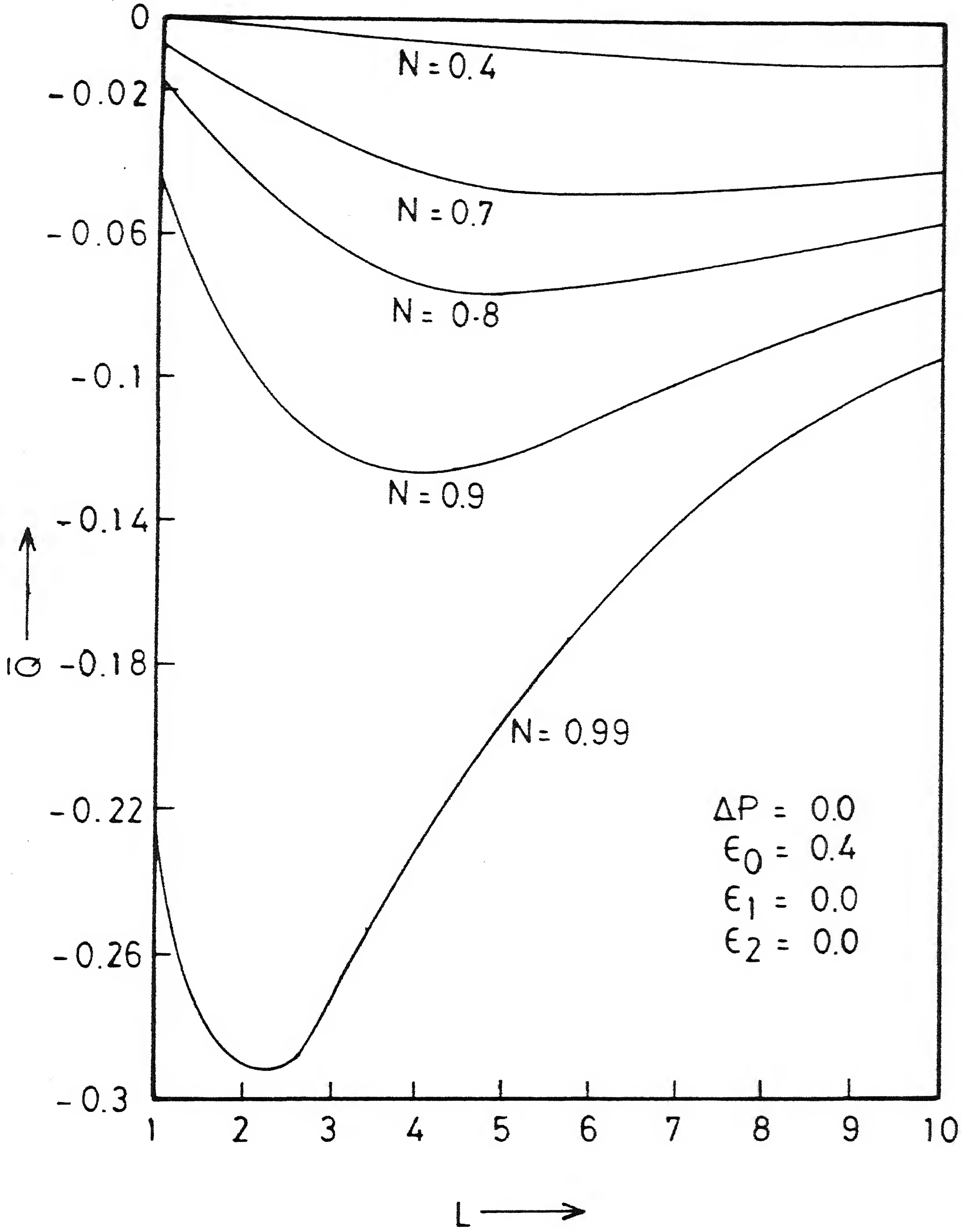


FIG.5.12 VARIATION OF $\bar{Q}$ WITH L FOR DIFFERENT N.

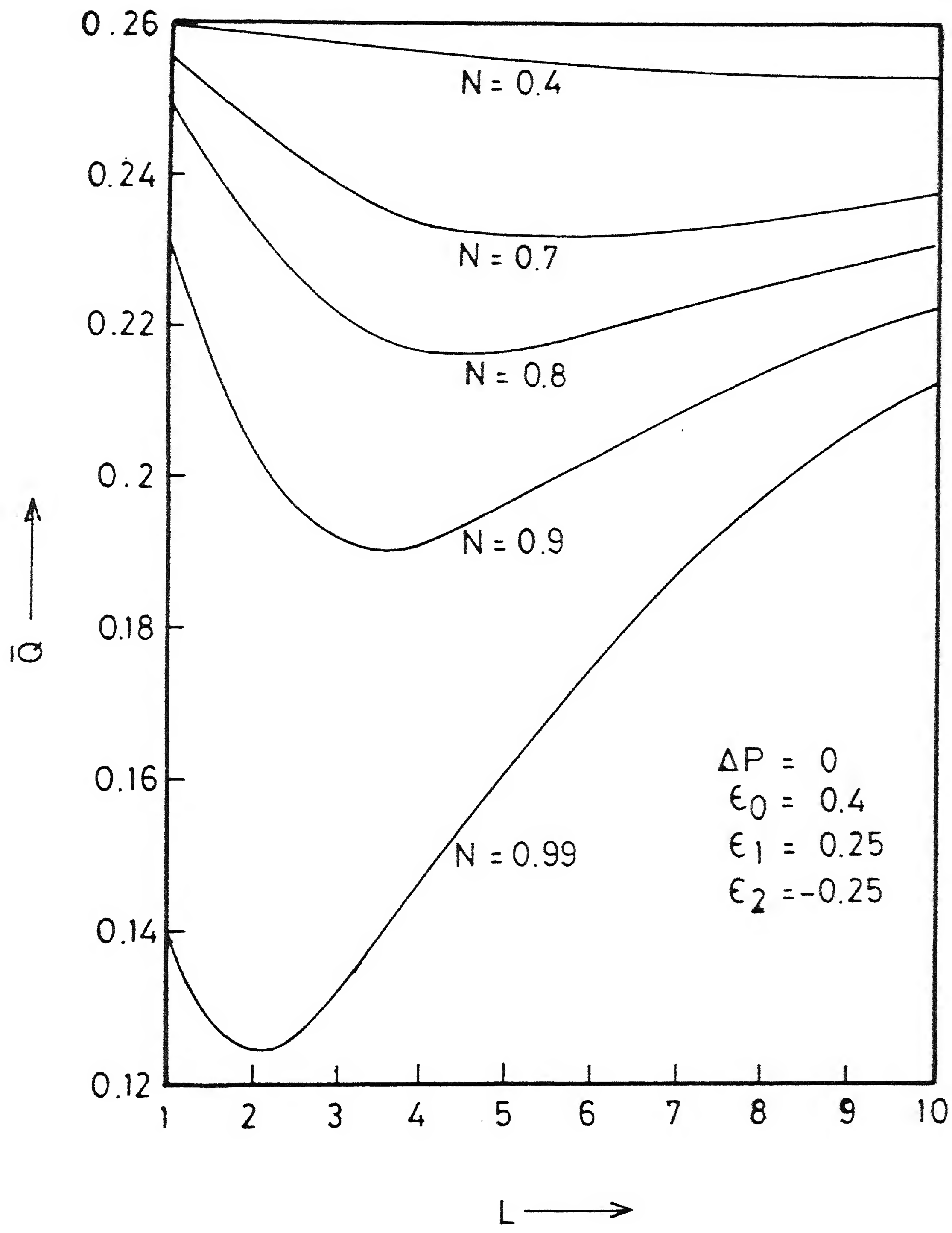


FIG.5.13 VARIATION OF $\bar{Q}$ WITH L FOR DIFFERENT N.

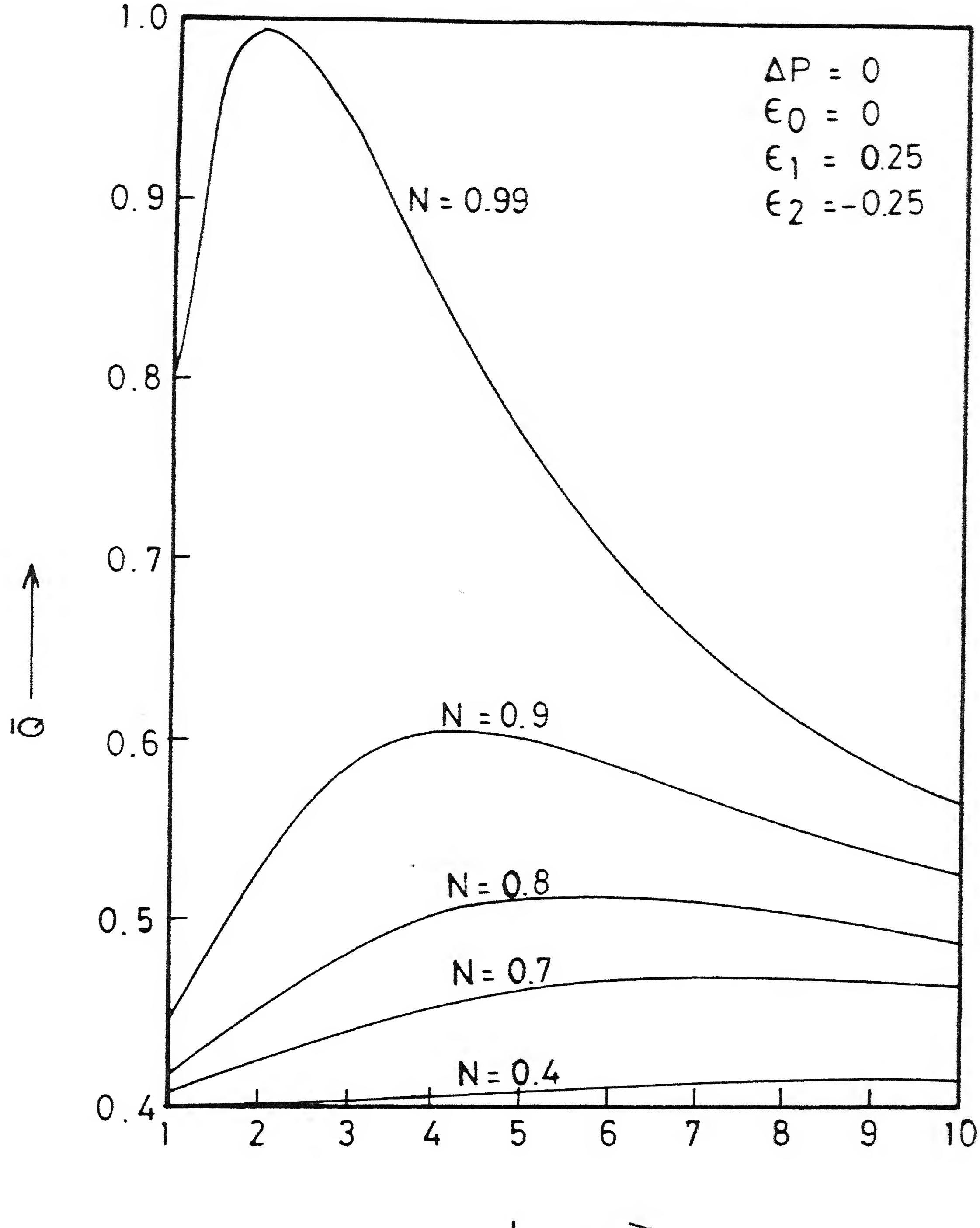


FIG.5.14 VARIATION OF $\bar{Q}$ WITH L FOR DIFFERENT N.

$$\bar{K}_i = \frac{K_i'}{h^2 \mu} \quad ; \quad V_p = \frac{V_p'}{C}$$

$$E_1 = -\frac{\zeta_1}{\mu} \quad ; \quad E_2 = \frac{\zeta_2}{\mu} \qquad \text{and} \qquad E_3 = \frac{K}{\mu}$$

the equations of motion of simple microfluid (eqns. 1.9 to 1.12) can be written in the non-dimensional form under long wavelength approximation ($h \ll \lambda$) as follows :

$$-\frac{\partial p^{\pm}}{\partial x} + \frac{\partial^2 u^{\pm}}{\partial z^2} - E_1 \frac{\partial \nu^{\pm}_{(13)}}{\partial z} + E_3 \left[\frac{\partial \nu^{\pm}_{[31]}}{\partial z} + \frac{1}{2}\frac{\partial^2 u^{\pm}}{\partial z^2}\right] = 0 \qquad (5.33)$$

$$\frac{\partial p^{\pm}}{\partial z} = 0 \qquad (5.34)$$

$$(\bar{K}_1-\bar{K}_3)\frac{\partial^2}{\partial z^2}(\nu^{\pm}_{31}) + (\bar{K}_2-\bar{K}_4)\frac{\partial^2}{\partial z^2}(\nu^{\pm}_{13}) - 2(E_1+2E_2)\,\nu^{\pm}_{(13)}$$

$$- E_3\left[2\nu^{\pm}_{[13]} + \frac{\partial u^{\pm}}{\partial z}\right] = 0 \qquad (5.35)$$

$$(\bar{K}_1+\bar{K}_3)\frac{\partial^2}{\partial z^2}(\nu^{\pm}_{31}) + (\bar{K}_2+\bar{K}_4)\frac{\partial^2}{\partial z^2}(\nu^{\pm}_{13}) - 2(E_1+2E_2)\,\nu^{\pm}_{(13)}$$

$$+ E_3\left[2\nu^{\pm}_{[13]} + \frac{\partial u^{\pm}}{\partial z}\right] = 0 \qquad (5.36)$$

where superscripts ($\pm$) denote the flow quantities in the region $h_o(x) \le z \le h_1(x)$ and $h_2(x) \le z \le h_o(x)$ respectively.

The boundary conditions are written in dimensionless form as

$$\left.\begin{array}{llll} u^{+} = u^{-} = 1 & & \text{at} & z = h_o(x) \\ u^{+} = V_{p-1} & & \text{at} & z = h_1(x) \\ u^{-} = V_{p-1} & & \text{at} & z = h_2(x) \\ \nu^{+}_{13} = 0 = \nu^{+}_{31} & & \text{at} & z = h_o(x),\ h_1(x) \\ \nu^{-}_{13} = 0 = \nu^{-}_{31} & & \text{at} & z = h_o(x),\ h_2(x). \end{array}\right\} \qquad (5.37)$$

The conditions of ν_{13} are perfect adherence conditions can be taken as the result of strong interaction of fluid and solid interface.

Further, using the stress-strain relationship and the long wavelength approximation the force equilibrium condition eq. (5.10) can be written as

$$\int_o^1 \left[\frac{\partial p}{\partial x} \right] dx = 0$$

$$\int_o^1 \left\{ \left(1 - \frac{E_3}{2}\right) \left[\frac{\partial u}{\partial z} \right] \Bigg|_{z=h_o(x)} + \frac{dh_o}{dx} \left[\frac{\partial p}{\partial x} \right] \right\} dx = 0 \qquad (5.38)$$

5.4.2 Analysis

Solving equations (5.34) to (5.36) for ν^{+}_{13} & ν^{+}_{31} and u^{+} alongwith appropriate boundary conditions (5.37) we get,

$$\nu^{+}_{(13)} = \sum_{i=1}^{2} a_i \left\{ \left[B_1 G_1 \ (a_i, \ a_{i+1}) + P^{+} G_3 \ (a_i, a_{i+1}) \right] \cosh(a_i z) \right.$$

$$\left. - \left[B_1 G_2 (a_i, a_{i+1}) + P^{+} G_4 (a_i, a_{i+1}) \right] \sinh(a_i z) \right\} \qquad (5.39)$$

$$\nu^{+}_{[13]} = - 2E_3 \frac{B_1 - zP}{(2+E_3)\ (\bar{K}_3 \alpha_3^2 - \bar{K}_4 \alpha_4^2)}$$

$$+ \sum_{i=1}^{2} a_i \left\{ \left[B_1 G_5 \ (a_i, a_{i+1}) + P^{+} G_7 \ (a_i, a_{i+1}) \right] \cosh a_i z \right.$$

$$\left. - \left[B_1 G_6 \ (a_i, a_{i+1}) + P^{+} G_8 \ (a_i, a_{i+1}) \right] \sinh a_i z \right\} \qquad (5.40)$$

$$u^{+} = -1 + V_p \left[D_1 \left\{ (z-h_o) + D_3(z) \right\} \right] + P^{+} \left[D_2 \left\{ (z-h_o) + D_3(z) \right\} \right.$$

$$\left. - \frac{z^2 - h_o^2}{2} + D_4(z) \right] \qquad (5.41)$$

where

$$D_1 = \frac{1}{(h_1 - h_o) + D_3(h_1)}$$

$$D_2 = D_1 \left[\frac{h_1^2 - h_o^2}{2} - D_4(h_1) \right]$$

$$D_3(z) = \frac{2}{2+E_3} \sum_i \Big[E_1\Big\{G_1(a_i,a_{i+1})\ L_1(a_iz) + G_2(a_i,a_{i+1})\ L_2(a_iz)\Big\}$$

$$- E_3\Big\{G_5(a_i,a_{i+1})\ L_1(a_iz) + G_6(a_i,a_{i+1})\ L_2(a_iz)\Big\}\Big]$$

$$D_4(z) = \frac{2}{2+E_3} \sum_i \Big[E_1\Big\{G_3(a_i,a_{i+1})\ L_1(a_iz) + G_4(a_i,a_{i+1})\ L_2(a_iz)\Big\}$$

$$- E_3\Big\{G_7(a_i,a_{i+1})\ L_1(a_iz) + G_8(a_i,a_{i+1})\ L_2(a_iz)\Big\}\Big]$$

$$G_j(a_i,a_{i+1}) = \frac{(-1)^{j+1}}{2} \Big\{ 1 + F_3(a_i)\Big\}\ F_5(j,a_i,a_{i+1})\ L_j(a_ih_1)$$

$$G_{4+j}(a_i,a_{i+1}) = \frac{(-1)^{j+1}}{2} \Big\{ 1 - F_3(a_i)\Big\}\ F_5(j,a_i,a_{i+1})\ L_j(a_ih_1)$$

$(j = 1,2,3\ \&\ 4)$

$$L_1(a_iz) = \sinh(a_iz) - \sinh(a_ih_o)$$

$$L_2(a_iz) = \cosh(a_iz) - \cosh(a_ih_o)$$

$$L_3(a_iz) = h_o \sinh(a_iz) - z \sinh(a_ih_o)$$

$$L_4(a_iz) = h_o \cosh(a_iz) - z \cosh(a_ih_o)$$

$$F_3(a_i) = \left[\bar{K}_1\bar{K}_4\ (a_i^2-\alpha_4^2) - \bar{K}_2\bar{K}_3\ (a_i^2-\alpha_2^2)\right] \Big/ \left[\ (\alpha_3^2-\alpha_1^2)\ \bar{K}_1\bar{K}_3\ \right]$$

$$F_4(a_i) = \left[1 + F_3(a_i)\right] \Big/ \left[\ \bar{K}_3\alpha_3^2 - \bar{K}_4\alpha_4^2\ \right]$$

$$F_5^*(a_i,a_{i+1}) = \frac{2\ E_3\ F_4(a_i)}{(2+E_3)\ \left\{F_3(a_i)-\ F_3(a_{i+1})\right\}}\ \frac{1}{\sinh\ [a_{i+1}(h_1-h_o)]}$$

$$F_5(j,a_i,a_{i+1}) = \begin{cases} F_5^*(a_{i+1},a_i) & j \text{ is odd} \\ \\ F_5^*(a_i,a_{i+1}) & j \text{ is even} \end{cases}$$

$a_1 = a_3 = \bar{\alpha}$, $a_2 = \bar{\beta}$ and $\bar{\alpha}^2$, $\bar{\beta}^2$ are the positive roots of the eqn. (2.38). B_1 is integration constant which is evaluated through the application of the boundary condition

$$u^+ = V_p - 1 \quad \text{at} \quad z = h_1.$$

The flux in the upper half plane is then given by

$$q^+ = V_p\left[D_1\left\{\frac{h_1^2-h_o^2}{2} - h_o(h_1-h_o)+\bar{D}_3\right\}\right] + P^+\left[D_2\left\{\frac{h_1^2-h_o^2}{2} - h_o(h_1-h_o)+\bar{D}_3\right\}\right.$$

$$\left. - \frac{h_1^3-h_o^3}{6} + \frac{h_o^2}{2}(h_1-h_o) + \bar{D}_4\right] + (h_o-h_1) \qquad (5.42)$$

$$= P_1V_p + P_2\frac{\partial p^+}{\partial x} + P_3 \quad \text{(say)} \qquad (5.43)$$

where

$$\bar{D}_3 = \frac{2}{2+E_3} \sum_i \frac{1}{a_i}\left[E_1\left\{G_1(a_i,a_{i+1})\ S(a_i) + G_2(a_i,a_{i+1})\ C(a_i)\right\}\right.$$

$$\left. - E_3\left\{G_5(a_i,a_{i+1})\ S(a_i) + G_6(a_i,a_{i+1})\ C(a_i)\right\}\right]$$

and
$$\bar{D}_4 = \frac{2}{2+E_3} \sum_i \frac{1}{a_i}\left[E_1\left\{G_3(a_i,a_{i+1})\ S(a_i) + G_4(a_i,a_{i+1})\ C(a_i)\right\}\right.$$

$$\left. - E_3\left\{G_7(a_i,a_{i+1})\ S(a_i) + G_8(a_i,a_{i+1})\ C(a_i)\right\}\right]$$

$$S(a_i) = L_2(a_i h_1) - a_i(h_1-h_o)\sinh(a_i h_o)$$

$$C(a_i) = L_1(a_i h_1) - a_i(h_1-h_o)\sinh(a_i h_o)$$

Similarly we can write the expression for u^- as

$$u^- = -1 + V_p\left[D_5\left\{(z-h_o) + D_7(z)\right\}\right] + P^-\left[D_6\left\{(z-h_o) + D_7(z)\right\}\right.$$

$$\left. - \frac{z^2-h_o^2}{2} + D_8(z)\right] \tag{5.44}$$

where

$$D_5 = \frac{1}{(h_2-h_o) + D_7(h_2)}$$

$$D_6 = D_5\left[\frac{h_2^2 - h_o^2}{2} - D_8(h_2)\right]$$

$$D_7(z) = \frac{2}{2+E_3} \sum_i \Big[E_1\Big\{\bar{G}_1(a_i,a_{i+1})\, L_1(a_i z) + \bar{G}_2(a_i,a_{i+1})\, L_2(a_i z)\Big\}$$

$$- E_3\Big\{\bar{G}_5(a_i,a_{i+1})\, L_1(a_i z) + \bar{G}_6(a_i,a_{i+1})\, L_2(a_i z)\Big\}\Big]$$

$$D_8(z) = \frac{2}{2+E_3} \sum_i \Big[E_1\Big\{\bar{G}_3(a_i,a_{i+1})\, L_1(a_i z) + \bar{G}_4(a_i,a_{i+1})\, L_2(a_i z)\Big\}$$

$$- E_3\Big\{\bar{G}_7(a_i,a_{i+1})\, L_1(a_i z) + \bar{G}_8(a_i,a_{i+1})\, L_2(a_i z)\Big\}\Big]$$

$$\bar{G}_j(a_i,a_{i+1}) = \frac{(-1)^{j+1}}{2a_i} \Big\{ 1 + F_3(a_i)\Big\}\, \bar{F}_5(j,a_i,a_{i+1})\, L_j(a_i h_2)$$

$$\bar{G}_{4+j}(a_i,a_{i+1}) = \frac{(-1)^{j+1}}{2a_i} \Big\{ 1 - F_3(a_i)\Big\}\, \bar{F}_5(j,a_i,a_{i+1})\, L_j(a_i h_2)$$

$$\bar{F}_5^*(a_i,a_{i+1}) = \frac{2E_3}{2+E_3}\, \frac{F_4(a_i)}{F_3(a_i)-F_3(a_{i+1})} \cdot \frac{1}{\sinh[a_{i+1}(h_2-h_o)]}$$

$$\bar{F}_5(j,a_i,a_{i+1}) = \begin{cases} \bar{F}_5^*(a_{i+1},a_i) & j \text{ is odd} \\ \bar{F}_5^*(a_i,a_{i+1}) & j \text{ is even} \end{cases}$$

which gives flux q^- as follows,

$$q^- = -V_p\left[D_5\left\{ \frac{h_2^2-h_o^2}{2} - h_o(h_2-h_o)+\bar{D}_7\right\}\right] + P^-\left[D_6\left\{\frac{h_2^2-h_o^2}{2} - h_o(h_2-h_o)+\bar{D}_7\right\}\right.$$

$$- \frac{h_2^3-h_o^3}{6} + \frac{h_o^2}{2} (h_2-h_o) + \bar{D}_8 \Bigg] + (h_2-h_o) \qquad (5.45)$$

$$= P_4 V_p + P_5 \frac{\partial p^-}{\partial x} + P_6 \quad \text{(say)} \qquad (5.46)$$

where

$$\bar{D}_7 = \frac{2}{2+E_3} \sum_i \frac{1}{a_i} \Bigg[E_1 \Big\{ \bar{G}_1(a_i, a_{i+1}) \bar{S}\,(a_i) + \bar{G}_2(a_i, a_{i+1}) \bar{C}\,(a_i) \Big\}$$

$$- E_3 \Big\{ \bar{G}_5(a_i, a_{i+1}) \bar{S}\,(a_i) + \bar{G}_6(a_i, a_{i+1}) \bar{C}\,(a_i) \Big\}$$

$$\bar{D}_8 = \frac{2}{2+E_3} \sum_i \frac{1}{a_i} \Bigg[E_1 \Big\{ \bar{G}_3(a_i, a_{i+1}) \bar{S}\,(a_i) + \bar{G}_4(a_i, a_{i+1}) \bar{C}\,(a_i) \Big\}$$

$$- E_3 \Big\{ \bar{G}_7(a_i, a_{i+1}) \bar{S}\,(a_i) + \bar{G}_8(a_i, a_{i+1}) \bar{C}\,(a_i) \Big\}$$

$$\bar{S}\,(a_i) = L_2(a_i h_2) - a_i(h_2-h_o) \sinh(a_i h_o)$$

$$\bar{C}\,(a_i) = L_1(a_i h_2) - a_i(h_2-h_o) \sinh(a_i h_o)$$

From equations (5.43) and (5.46) we get

$$\frac{\partial p^+}{\partial x} = (q^+ - V_p\, P_1 - P_3)/P_2 \qquad (5.47)$$

$$\frac{\partial p^-}{\partial x} = (q^- - V_p\, P_4 - P_6)/P_5 \qquad (5.48)$$

Also we know that

$$\int_0^1 \frac{\partial p^+}{\partial x}\, dx = \Delta p = \int_0^1 \frac{\partial p^-}{\partial x}\, dx$$

i.e. the pressure rise over a wavelength is same for the two regions. We get from equations (5.47) and (5.48):

$$\Delta p = q^+ I_{11} + V_p I_{12} + I_{13} \tag{5.49}$$

$$\Delta p = q^- I_{21} + V_p I_{22} + I_{23} \tag{5.50}$$

where

$$I_{11} = \int_0^1 \frac{1}{P_2}\, dx \quad ; \quad I_{21} = \int_0^1 \frac{1}{P_5}\, dx$$

$$I_{12} = -\int_0^1 \frac{P_1}{P_2}\, dx \quad ; \quad I_{22} = -\int_0^1 \frac{P_4}{P_5}\, dx \tag{5.51}$$

$$I_{13} = -\int_0^1 \frac{P_3}{P_2}\, dx \; ; \quad I_{21} = -\int_0^1 \frac{P_6}{P_5}\, dx$$

Now considering force equilibrium condition (5.38) and substituting for $\frac{\partial p^+}{\partial x}$ and $\frac{\partial p^-}{\partial x}$ we get

$$q^+ I_{31} + I_{32}\, q^- + I_{33} V_p + I_{34} = 0 \tag{5.52}$$

where

$$I_{31} = \int_0^1 \left(\frac{P_8}{P_2} \right) dx \qquad I_{33} = \int_0^1 \left(P_7 - \frac{P_1 P_8}{P_2} - \frac{P_9 P_4}{P_5} \right) dx$$

$$I_{32} = \int_0^1 \left(\frac{P_9}{P_5} \right) dx \qquad I_{34} = - \int_0^1 \left(\frac{P_8 P_3}{P_2} + P_9 \frac{P_6}{P_5} \right) dx$$

$$P_7 = (D_1 - D_5) + D_1 \frac{\partial}{\partial z} \left[D_3(z) \right] - D_5 \frac{\partial}{\partial z} \left[D_7(z) \right]$$

$$P_8 = D_2 \left\{ 1 + \frac{\partial}{\partial z} \left[D_3(z) \right] \right\} + \frac{\partial}{\partial z} \left[D_4(z) \right]$$

$$P_9 = D_6 \left\{ 1 + \frac{\partial}{\partial z} \left[D_7(z) \right] \right\} + \frac{\partial}{\partial z} \left[D_8(z) \right]$$

Recalling that the time averaged flux $\bar{Q}$ is given by,

$$\bar{Q} = q + 2\,(1 - V_p)\,, \tag{5.53}$$

eliminating q^+, q^- and $q = q^+ + q^-$ from equations (5.52) and (5.53) we get the expression for V_p in terms of Δp. These quantities have been numerically calculated.

5.5 RESULTS AND DISCUSSION

The parameters involved in this analysis are : the simple microfluid parameter E_1, E_2, E_3, the wave amplitudes ϵ_1 and ϵ_2 and the pressure difference across the wavelength (Δp). Here the values of V_p and $\bar{Q}$ have been calculated numerically for $\bar{K}_1 = 1.1$,

$\bar{K}_2$=1.0, $\bar{K}_3$=-0.9 & $\bar{K}_4$=0.6 and are plotted versus ϵ_o in figures (5.15 to 5.21) and (5.22 to 5.24) respectively. Figures (5.15 to 5.17) show the effect of simple microfluid parameters when there is no transverse waves travelling along the channel walls. As in the previous case it is observed that V_p increases with the magnitude of ϵ_o, as also with Δp. In case of zero pressure difference across the wavelength, i.e. Δp = 0.0, V_p increases with the decrease in E_1, E_2 and increase in E_3. However, when Δp = -0.05 though it shows similar behaviour for higher values for $|\epsilon_o|$ reverse trend is observed for small $|\epsilon_o|$. It may be further remarked that unlike in the other chapters effect of E_2 variation is significant here.

Figures (5.18 to 5.21) show the effect of transverse motion of the channel walls on V_p. Even in the presence of transverse wave motion, V_p decreases as Δp decreases, and the effect of simple microfluid parameters remains same as has been observed earlier (ϵ_1 = 0 = ϵ_2) (Figs. 5.18 & 5.19). Moreover variation of V_p versus ϵ_o loses its symmetrical behaviour for non-symmetrical wave motion on the wall (Figs. 5.20 and 5.21). For given ϵ_1=0.35 amd Δp = -0.05, V_p increases considerably as ϵ_2 increases from 0.0 to 0.25.

The figures (5.22 to 5.24) show that $\bar{Q}$ increases as E_3 increases but it decreases with an increase in E_1 and E_2 parameters. Further, for given ϵ_2 = -0.25, ϵ_o = 0.0, $\bar{Q}$ decreases as ϵ_1 increases from 0.0 to 0.35 and a reverse trend is observed at ϵ_o = 0.25. In the case of ϵ_1 = 0.35 and ϵ_2 = -0.25, $\bar{Q}$

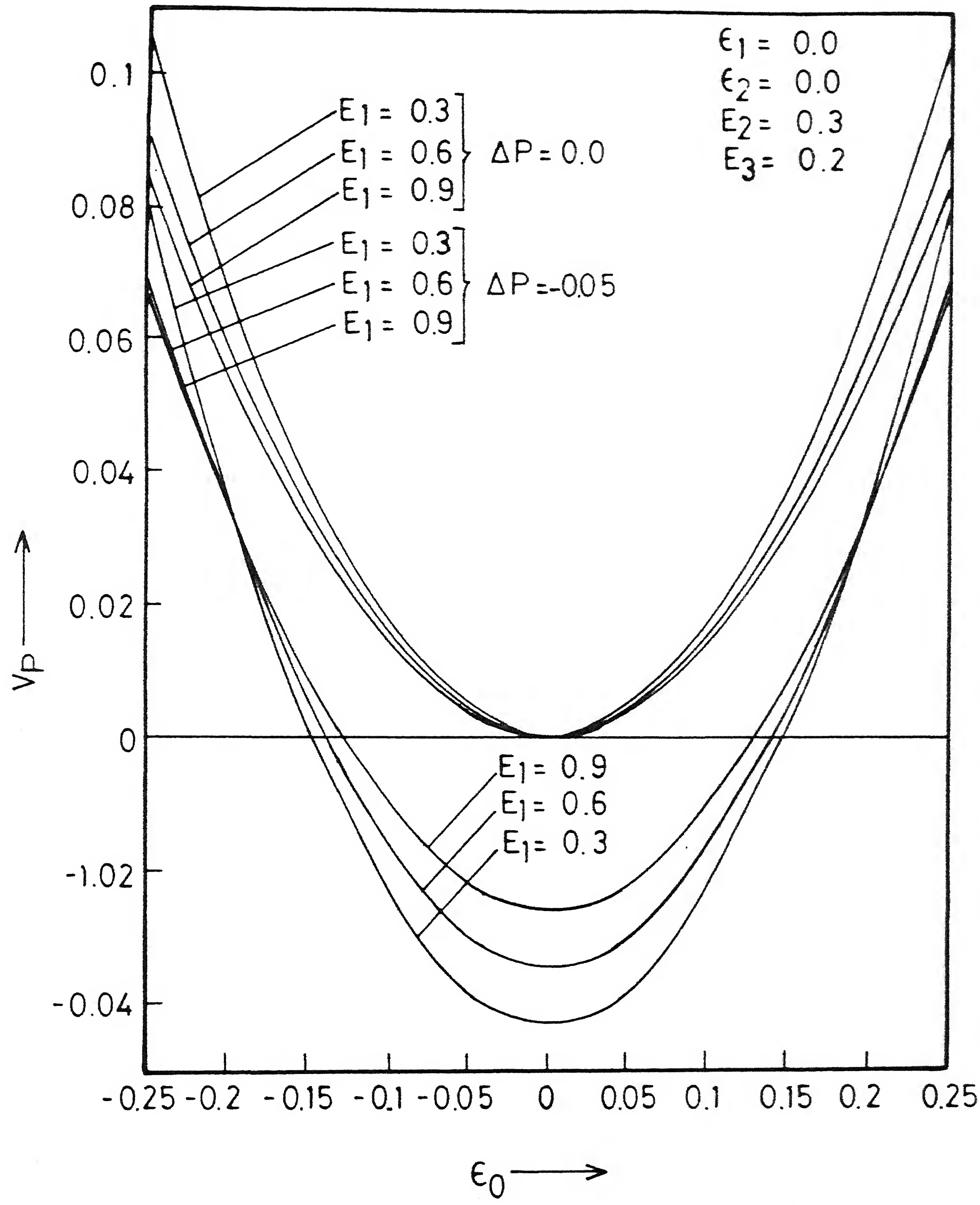


FIG.5.15 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT ΔP & E_1.

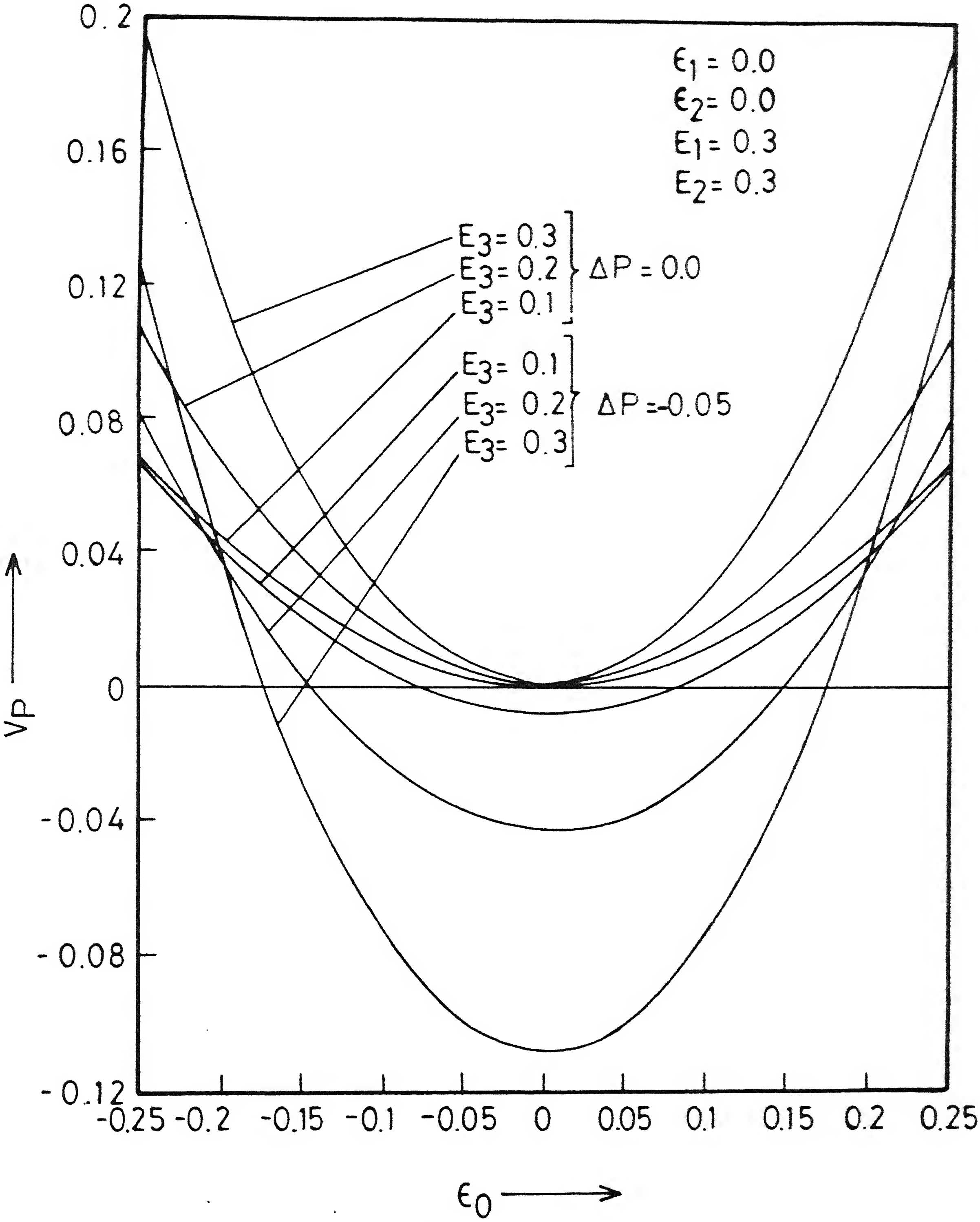


FIG.5.16 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT ΔP & E_3.

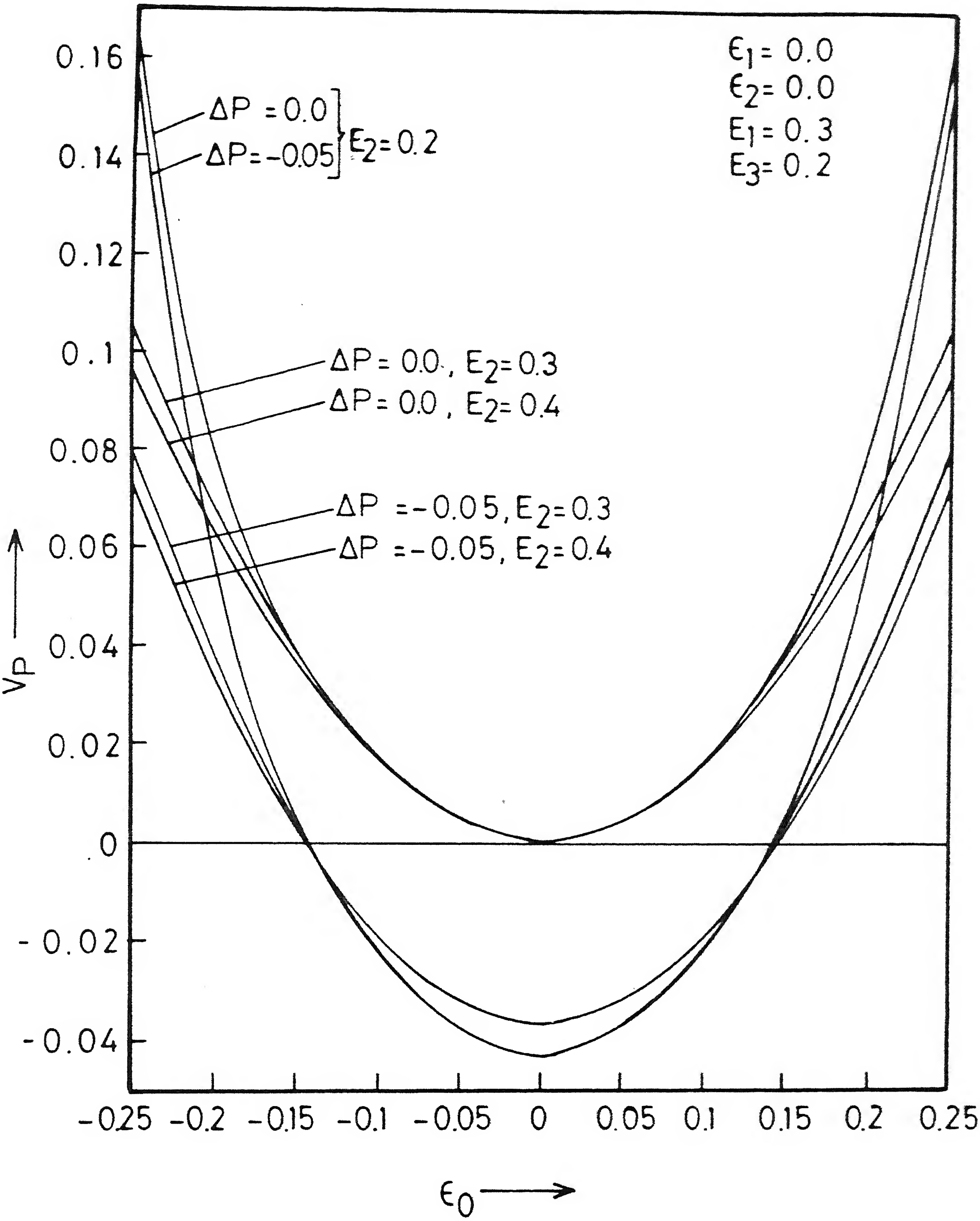


FIG. 5.17 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT ΔP & E_2.

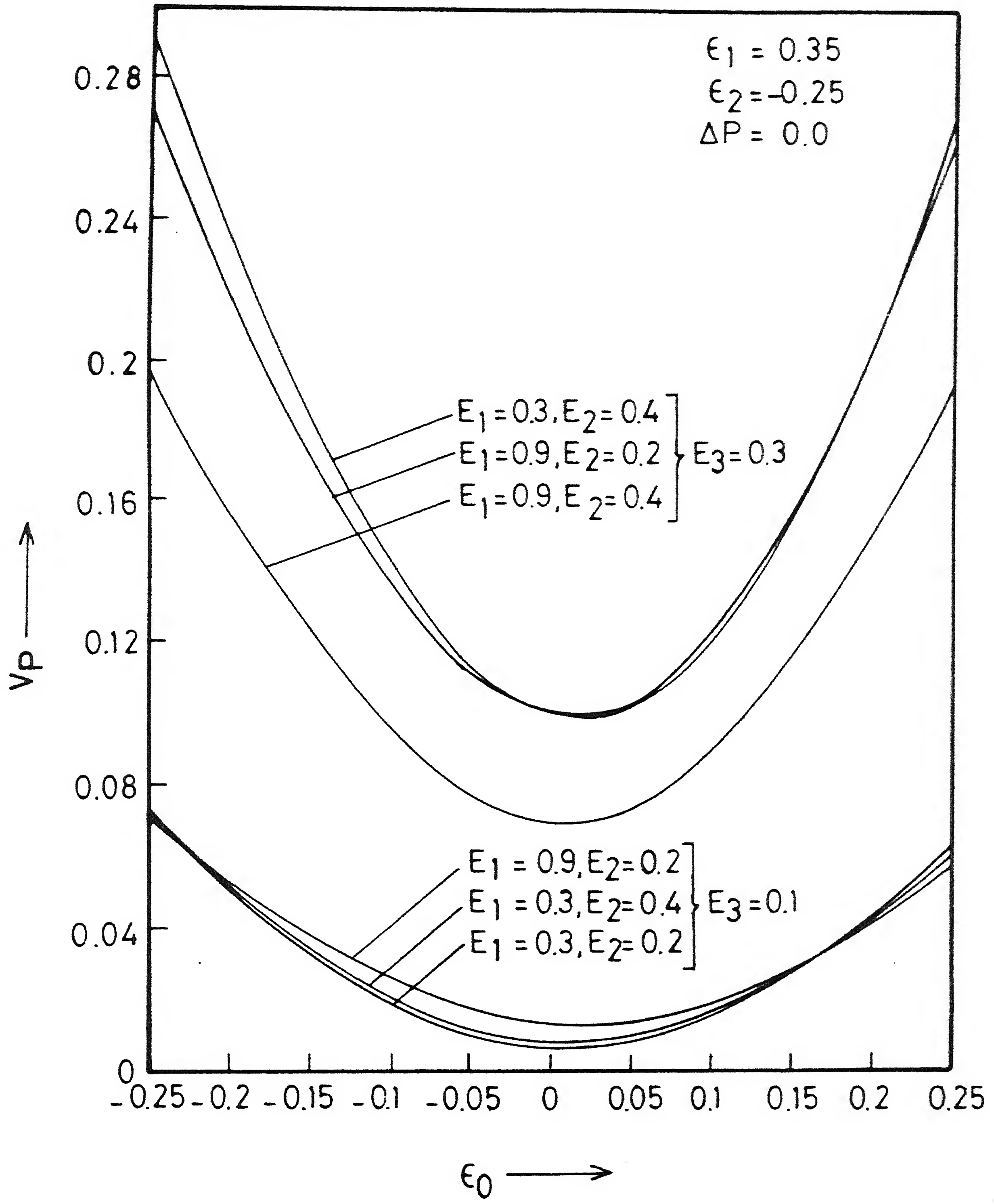


FIG. 5.18 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

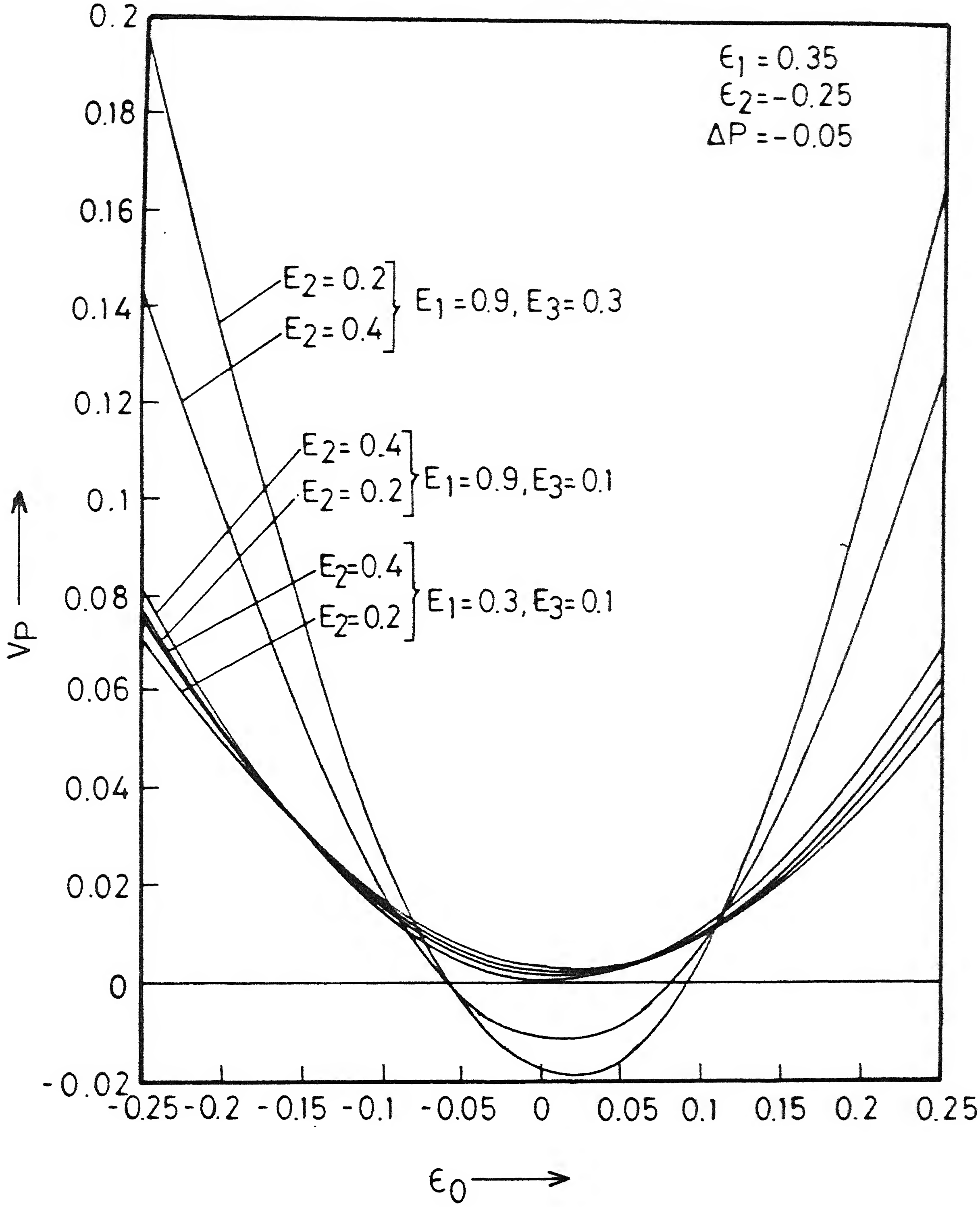


FIG.5.19 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

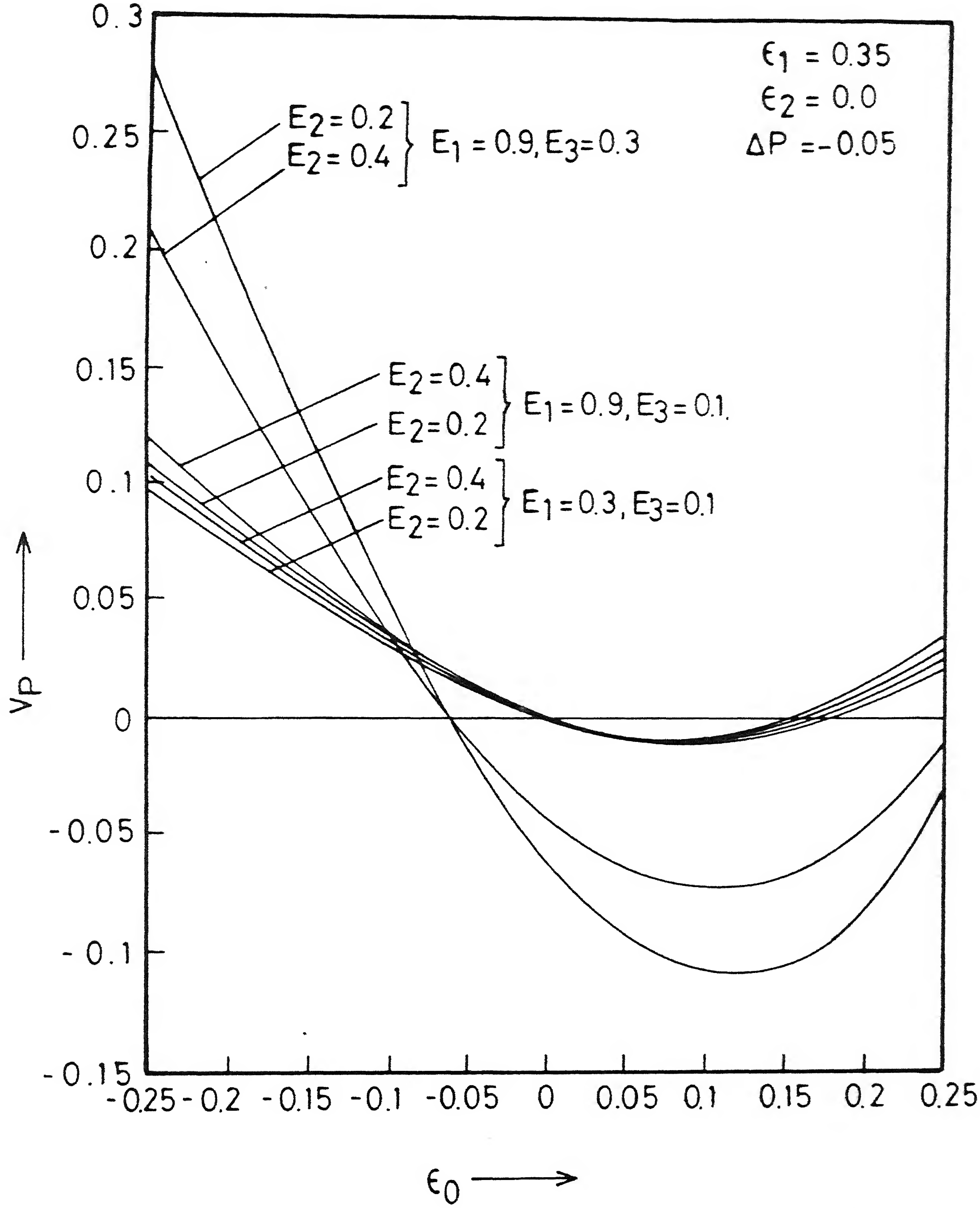


FIG.5.20 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

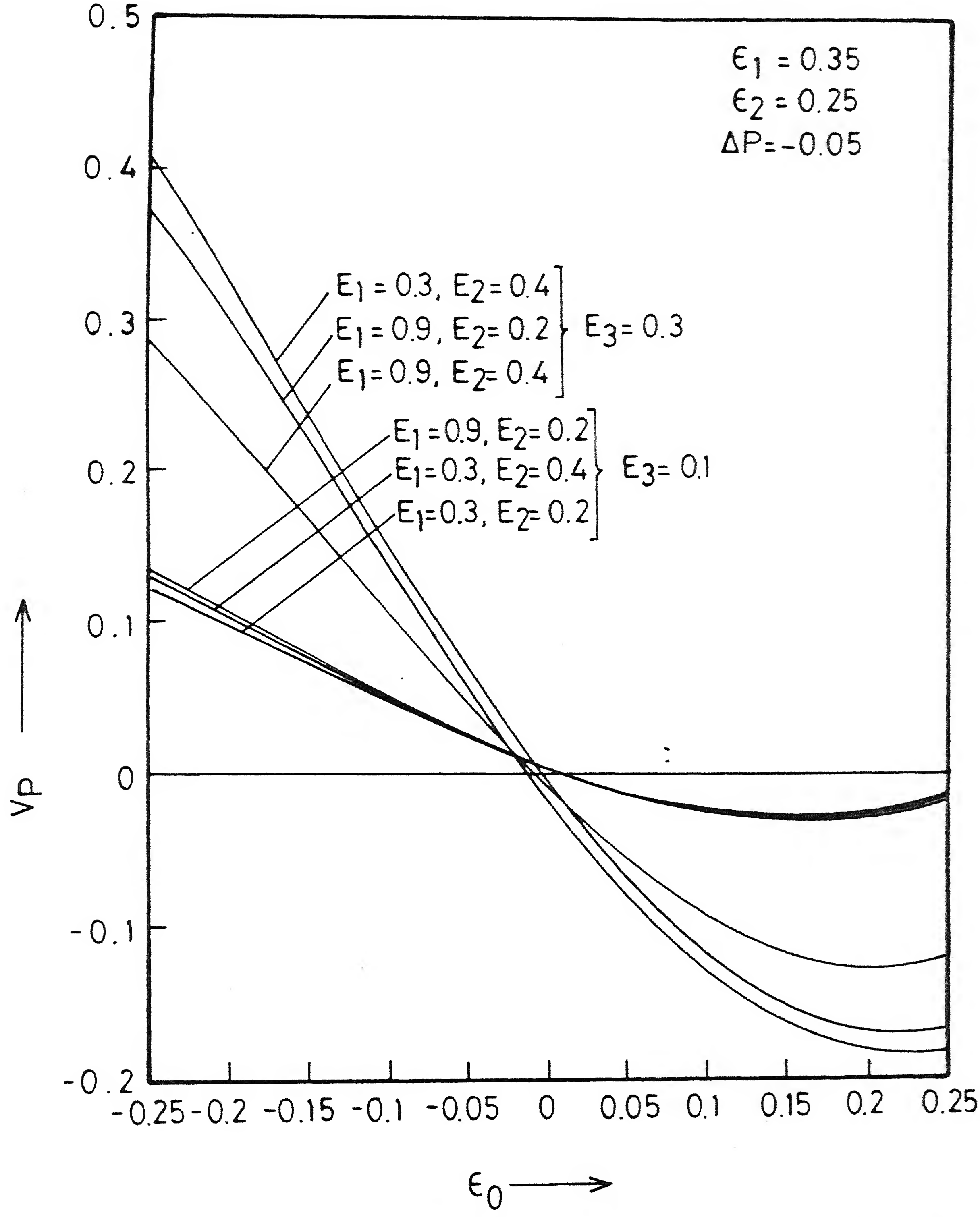


FIG. 5.21 VARIATION OF V_P WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

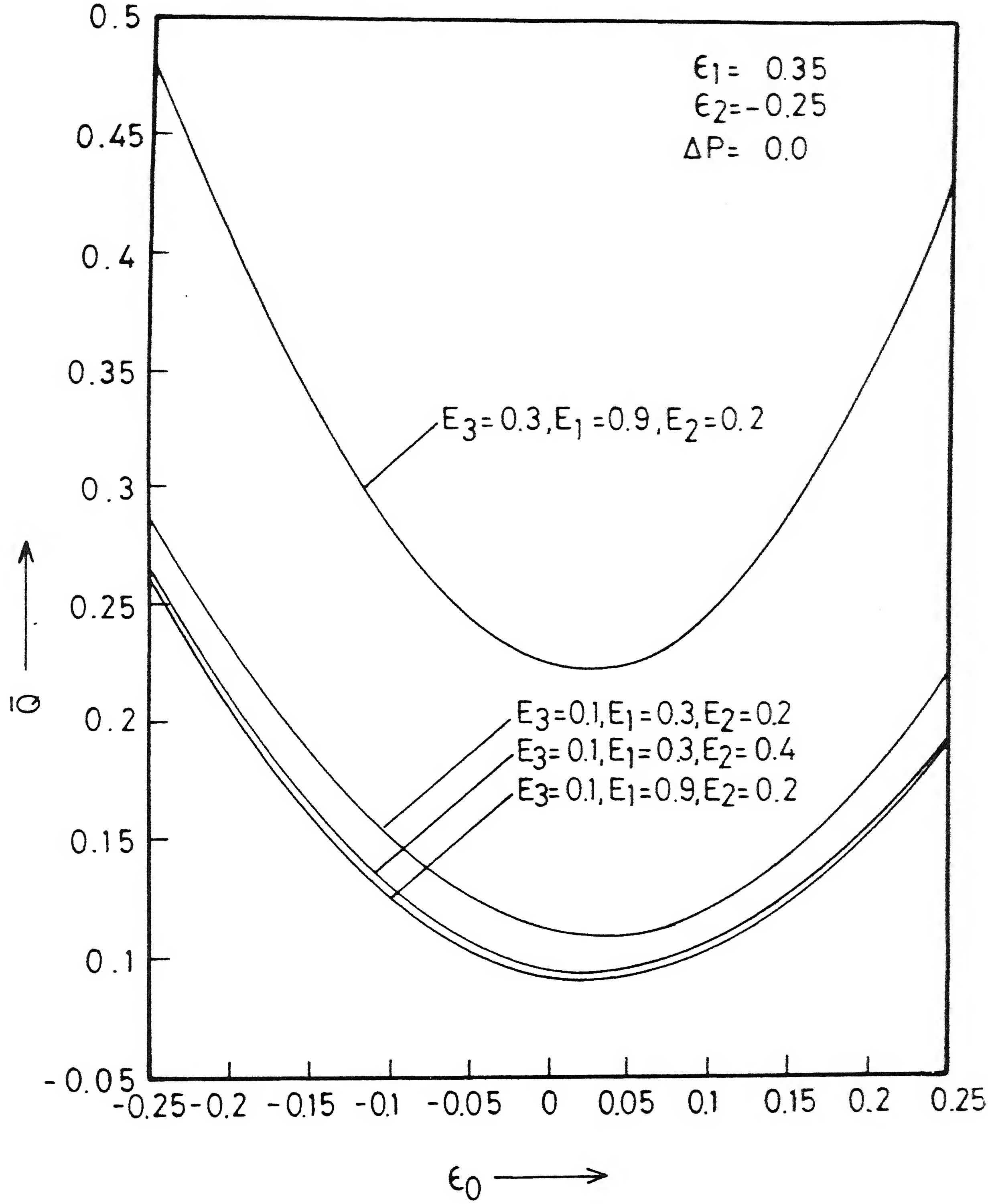


FIG. 5.22 VARIATION OF $\bar{Q}$ WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

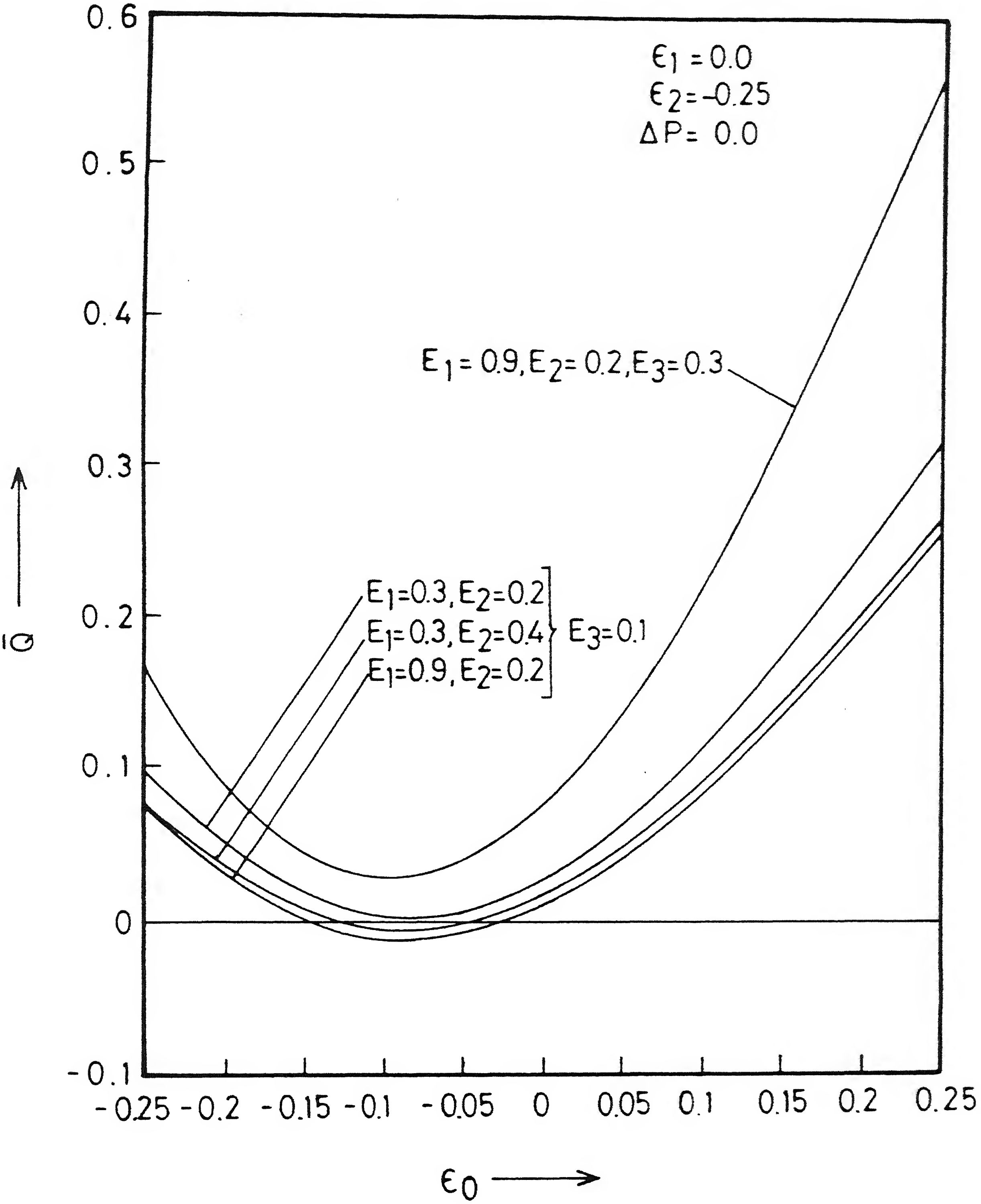


FIG. 5.23 VARIATION OF $\bar{Q}$ WITH ϵ_0 FOR DIFFERENT E_1, E_2 & E_3.

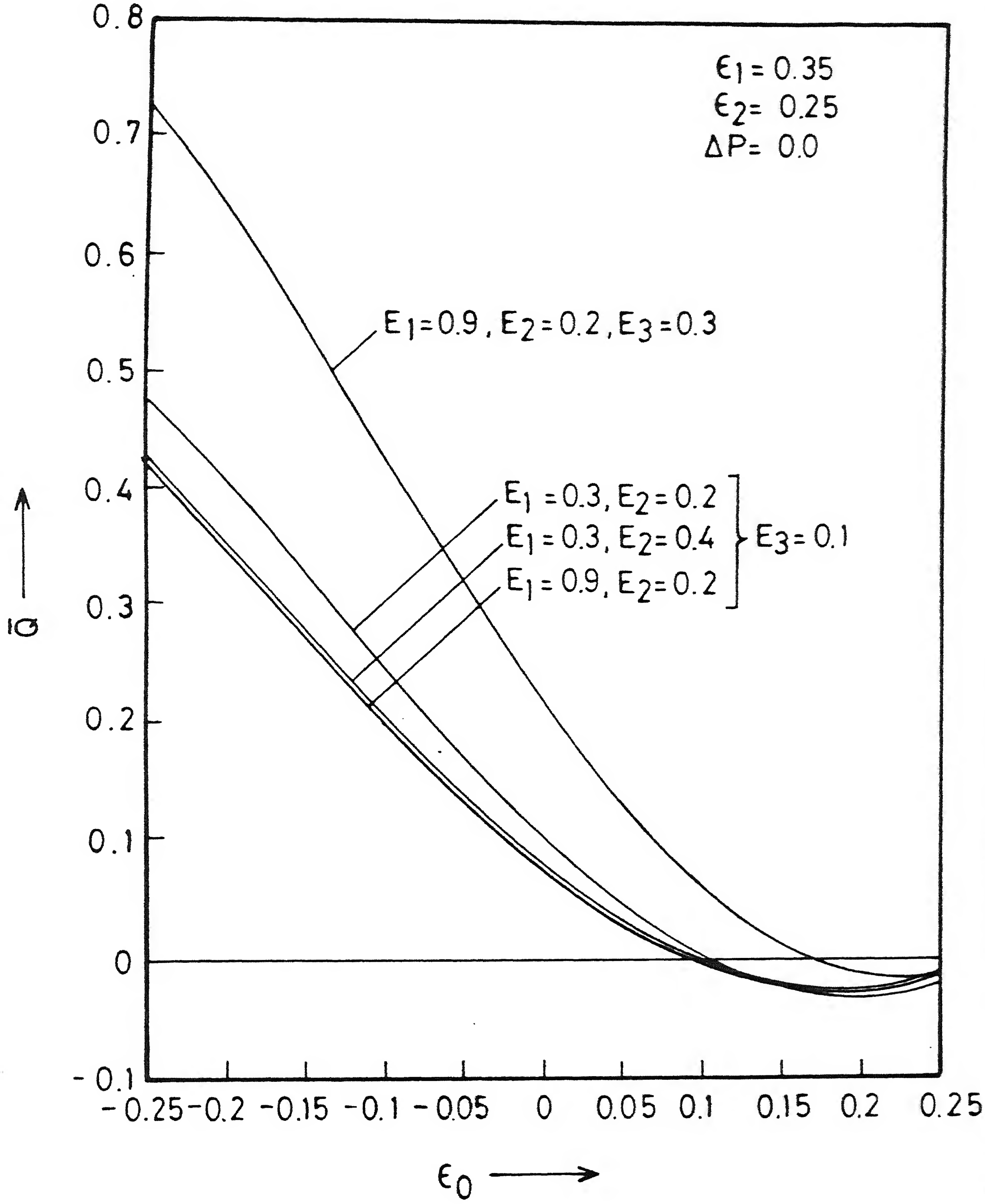


FIG.5.24 VARIATION OF $\bar{Q}$ WITH ϵ_0 FOR DIFFERENT E_1,E_2 & E_3

initially decrease with ϵ_o and then shows sharp increase, while in the case of $\epsilon_1 = 0.35$ and $\epsilon_2 = 0.25$ it shows sharp decrease as ϵ_o increases from-0.25 to 0.15 and then show a slight increase.

5.6 CONCLUSION

In the present study, a microcontinuum approach has been taken to model the mucus as micropolar fluid (in part I) and as simple microfluid (in part II). It is observed that the propulsive velocity increases as the departure from Newtonian theory increases. For simple microfluid model, V_p increases with the decrease in microstretching parameters and increase in rotation parameter. In both the cases propulsive velocity increases with amplitude of the wave on the sheet. Peristaltic wave on the wall of channel decreases the propulsive velocity. This suggests that if by some biochemical means, a peristaltic wave can be imposed on the walls of the cervical canal, then the propulsive velocity of the spermatozoa can be reduced considerably, so that it will not reach the point of fertilization. This observation is important from the point of view of fertility control.

REFERENCES

AERO, E.L., BULYGIN, A.N., KUVSHINSKII, E.V. (1965)
Asymmetric Hydromechanics.
PMM, 29, 297-308.

ALLEN, S.J., DeSILVA, C.N. AND KLINE, K.A. (1967).
A theory of simple deformable directed fluids.
Phys. of Fluids, 10, 2551.

ALLEN, S.J. AND KLINE, K.A. (1969).
A Theory of Mixtures with Microstructure.
J. of Appl. Math. and Phys. (ZAMP), 20, 145-155.

ARIMAN, T. (1971).
On the analysis of blood flow .
J. Biomech., 4, 185.

ARIMAN, T., TURK, M.A. AND SYLVESTER, N.D. (1973a).
Microcontinuum fluid mechanics - A Review.
Int. J. Engng. Sci., 11, 905-930.

ARIMAN, T., TURK, M.A. AND SYLVESTER, N.D. (1973b).
A Microcontinuum Model of Blood with deformable cells,
ASME Biomechanics Symposium (Edited by Y.C. Fung and J.A. Brighton).

ARIMAN, T., TURK, M.A. AND SYLVESTER, N.D. (1974a).
On steady and pulsatile flow of blood.
J. Appl. Mech. Trans. ASME, 41, 1-7.

ARIMAN, T., TURK, M.A. AND SYLVESTER, N.D. (1974b).
Applications of microcontinuum fluid mechanics.
Int.J. Engng. Sci., 12, 273-293.

ARIS, R. (1956).
On the dispersion of a solute in a fluid flowing through a tube.
Proc. Roy. Soc. Lond., A235, 67-77.

BARTON, N.G. (1984).
J. Austra. Math. Soc., B25, 287.

BARTON, C. AND RAYNOR, S. (1968).
Peristaltic flow in tubes.
Bull. Math. Biophys., 30, 663-680.

BIRD, R.B., STEWARD, W.E. AND LIGHTFOOT, E.N. (1960).
Transport phenomena, New York, John Wiley and Sons.

BLAKE, J.R., VANN, P.G. AND WINET, H. (1983).
A model of Ovum Transport.
J. Theor. Bio., 102, 145-166.

BÖHME, G. AND FRIEDRICH, R. (1983).
Peristaltic flow of viscoelastic liquids.
J. Fl. Mech., 128, 109-122.

BUGLIARELLO, G. AND SEVILLA, J. (1970).
Velocity Distribution and other characteristics of steady and Pulsatile Blood flow in Fine glass Tubes.
Biorheology, 7, 85-107.

BURNS, J.C. AND PARKES, T. (1967).
Peristaltic Motion.
J. Fl. Mech., 29, Part 4, 731-743.

CARO, C.G., FITZ-ZERALD, J.M. AND SCHROTER, R.C. (1971).
Atheroma and arterial wall shear : observation, correlation and proposal of a shear dependent mass transfer mechanism for atherogenesis.
Proc. Roy. Soc. Lond., B177, 109-159.

CHANDRA, P. (1980).
Nat. Acad. Sci. India:Golden Jubilee Commemoration,359.

CHANDRA, P. AND AGRAWAL, R.P. (1983).
Dispersion in Simple microfluid flows.
Int. J. Engng. Sci., 21, 5, 431-441.

CHARM, S.E., KURLAND, G.S. AND BROWN, S.L. (1968).
The influence of Radial Distribution and Marginal Plasma Layer or the Flow of Red Cell Suspensions.
Biorheology, 5, 15-43.

CHATURANI, P. AND SAMY R.P. (1983).
Blood flow through stenosed arteries.
Proc. of the First Int. Con. of Physio. Fluid Dynamics, Madras, India (Sept. 5-7), (Costed and Scpfd, India).
(Eds) Nigam, S.D. and Singh, M., Page 63-67.

CHENG, L.C., CLARK, M.E. AND ROBERTSON, J. M. (1972).
Numerical calculation of oscillating flow in the vicinity of square wall obstacles in plane conduits.
J. Biomech., 5, 567-584.

CHENG, L.C., ROBERTSON, J.M. AND CLARK, M.E. (1973).
Numerical calculation of plane oscillatory pressure gradient and frequency with square wall obstacles.
J. Biomech., 6, 521-538.

CHOW, T.S. (1970).
Peristaltic transport with circular cylindrical pipe.
J. Appl. Mech., 37, 901-905.

COKELET, G.R. (1972).
The rheology of human blood.
In : Y.C. Fung (ed.), Biomechanics : Its foundation and objectives,p.63, Prentice Hall (Englewood Cliffs, N.J).

COWIN, S.C. (1972).
On the polar flud as a model for blood flow in tubes.
Biorheology, 9, 23-25.

CUMMING, G., CRANK, J., HORSFIELD, K. AND PARKER, I. (1966).
Gaseous diffusion in airways of human lung.
Resp. Physio., 1, 58-74.

DALY, B.J. (1976).
A numerical study of pulsatile flow through stenosed canine femoral arteries.
J.Biomech., 9, 465-475.

DAS, S.K. AND MAZUMDER, B.S. (1989).
Dispersion of reactive solute in liquid flowing through a tube.
Int. J. Engng. Sci., 27, 10, 1203-1209.

DAVAJAN, V., NAKAMURA, R.M. AND MISHELL, D.R. (1971).
A simplified technique for evaluation of biophysical properties of cervical mucus.
Am. J. Obstet. Gynec, 109, 1042-1048.

DESHPANDE, M.D., GIDDENS, D.P. AND MABON, R.F. (1976).
Steady laminar flow through modelled vascular stenoses.
J. Biomech., 9, 165-174.

DEVI, R.G. AND DEVANATHAN, R. (1975).
Peristaltic motion of a micropolar fluid.
The Proc. of the Ind. Acad. Sci., 81, Sec.A, 149-163.

DUTTA, B.K., ROY, N.C. AND GUPTA, A.S. (1974).
Dispersion of a solute in a non-Newtonian fluid with simulation chemical reaction. Studia, Universitatis Babes-Bolyai,Series Mathematica-mechanica, CLUJ, 78-83.

ELSTEIN, M., MITCHELL, R.F. AND SYRETT, J.T. (1971).
Ultrastructure of cervical mucus.
J. Obstet. Gynec. Brit. Comm., 78, 180-183.

ERINGEN, A.C. (1964).
Simple Microfluids.
Int. J. Engng. Sci., 2, 205-217.

ERINGEN, A.C. (1966).
Theory of micropolar fluids.
J. Math. Mech., 16, 1-18.

ERINGEN, A.C. AND SUHUBI, E.S. (1964).
Nonlinear theory of simple micro-elastic solids.
Int. J. Engng. Sci., 2, 189-203.

FAN, L.T. AND HWANG, W.S. (1965).
Dispersion of ostwald - de - wade fluid in laminar flow through a cylindrical tube.
Proc. Roy. Soc. Lond., A283, 576-582.

FAN, L.T. AND WANG, C.B. (1966).
Dispersion of matter in non-Newtonian laminar flow through circular tubes.
Proc. Roy. Soc. Lond., A292, 203-208.

FORRESTER, J.F. AND YOUNG, D.F. (1970).
Flow through a converging and diverging tube and its implication in oclusive vascular disease.
J. Biomech., 3, 297-316.

FUNG, Y.C. AND YIH, C.S. (1968).
Peristaltic transport.
J. Appl. Mech., 35, 669-675.

GHOSAL, S. (1971).
Dispersion of solutes in non-Newtonian flows through a circular tube.
Chem. Engg. Sci., 26, 185-188.

GILL, W.N. (1967).
A note on the solution of transient dispersion problem.
Proc. Roy. Soc. Lond., 298A, 335-339.

GILL, W.N. AND SANKARA SUBRAMANIAN, R. (1970).
Exact analysis of unsteady convective diffusion.
Proc. of Roy. Soc. Lond., 316A, 341-350.

GOLDSMITH, H.L. AND SKALAK, R. (1975).
Hemodynamics : in M. Van. Dyke (Ed.), Annual review of fluid dynamics, Vol. 7, Annual Reviews Incorporated (Palo Alto, California, USA).

GOPALAN, N.P. AND DEVANATHAN, R. (1983).
Pressure distribution in peristaltic transport of fluids with an inserted catheler - A model of urometrogram.
Proc. of the First Int. Con. of Physio. Fluid Dynamics, Madras, India (Sept. 5-7), (Costed and Scpfd, India).
(Eds) Nigam, S.D. and Singh, M., Page 13-16.

GRAY, J. AND HANCOCK, G.J. (1955).
The propulsion of sea urchin spermatozoa.
J. Exp. Biol., 32, 802-814.

GUPTA, P.S. AND GUPTA, A.S. (1972).
Effects of homogeneous and heterogeneous reaction on the dispersion of a solute in the laminar flow between two plates.
Proc. Roy. Soc. Lond., A330, 59-63.

GUPTA, B.B. AND SESHADRI, V. (1976).
Peristaltic pumping in non-uniform tubes.
J. Biomech., 9, 105-109.

HANCOCK, G.J.(1953).
The self propulsion of microscopic organisms through liquids.
Proc. Roy. Soc. Lond., A217, 96-121.

HAN, C.D. AND BARNETT, B. (1973).
Measrements of the rheological properties of biological fluids. In : Rheology of Biological Systems (Edited by L. Gabelnick and Mitchell Litt), Illinois : Charles C. Thomas Publisher, 195-217.

HERSHEY, D. AND CHO S.J. (1966).
Blood flow in rigid tubes : thickness and slip velocity of plasma film at the wall.
J. Appl. Physiol., 21, 27-32.

HOUSSAY, B.A. (1955).
Human Physiology. McGraw-Hill.

JAFFRIN, M.Y. AND SHAPIRO, A.H. (1971).
Peristaltic Pumping.
Ann. Rev. Fluid Mech., 3, 13-26.

KALONI, P.N. AND DeSILVA, C.N. (1969).
Oriented fluids and the rheology of suspension.
Phys. of Fluid, 12, 994-999.

KAIMAL, M.R. (1978).
Peristaltic pumping of a Newtonian fluid with particles suspended in it at low Reynolds number under long wavelength approximation.
J. Appl. Mech. ASME Trans., 43, 32-36.

KANG, C.K. AND ERINGEN, A.C. (1976).
The effect of microstructure on the rheological properties of blood.
Bull. Math. Biology, 38, 135-159.

KARTZ, S. (1959).
Chemical reaction catalysed on a tube wall.
Chem. Engg. Sci., 10, 202-211.

KATZ, D.F. (1974).
On the propulsion of micro-organisms near solid boundaries.
J. of Fluid Mech., 64, 33-50.

KIRWAN, JR. A.D. (1986).
Boundary conditions for micropolar fluids.
Lett. Appl. Engng. Sci., 24, 1237-1242.

LEE, J.S. AND FUNG, Y.C. (1970).
Flow in a locally constricted tube at low Reynolds numbers.
J. Apply. Mech., 37, 9-16.

Liepsch, D.W. (1987).
The role of biofluidmechanics in atherosclerosis.
Proc. of Second Int. Conf. Physio. Fl. Dyn., IIT Madras (August 10-12), (Eds Srinath L.S. and Singh, M). Tata McGraw Hill Pulishing Company Limited, New Delhi, Page 26-33.

LIH, C.H. (1970).
Peristaltic transport in circular cylindrical tube.
J. Biomech., 3, 513-523.

LIGHTHILL, M.J. (1974).
Mathematical Bio-Fluid Dynamics, SIAM, USA.

LIGHTHILL, M.J. (1975).
Mathematics of aquatic animal locomotion at low Reynolds number.
Winter School in Physiological Fluid Dynamics, I.I.T., Delhi, India, 8-18 December.

LIH, M.M. (1975).
Transport phenomena in medicine and biology.
New York, Wiley.

LIRON, N. (1976).
On peristaltic flow and its efficiency.
Bull. of Math. Bio., 38, 573-596.

LYKOUDIS, P.S. AND ROOS, R. (1970).
The fluid mechanics of the ureter from a lubrication point of view.
J. Fl. Mech., 43, 661-674.

MANN, F.C., HERRICK, J.F., ESSEX, H.E. AND BALDES, E.J. (1938).
Effect on blood flow of decreasing the lumen of a blood vessel.
Surgery, 4, 249-252.

MANTON, M.J. (1975).
Long wavelength peristaltic pumping at low Reynolds number.
J. Fl. Mech., 68, 467-476.

McDONALD, D.A. (1979).
On steady flow through modelled vascular stenosis.
J. Biomech., 12, 13-20.

MIDDLEMAN, S. (1972).
Transport phenomena in the cardio vascular system.
New York, Wiley.

MIGUN, N.P. (1984).
On hydrodynamic boundary conditions for microstructural fluids.
Rheol. Acta., 23, 575-581.

MITTRA, T.K. AND PRASAD, S.N. (1974).
Interaction of peristaltic motion with poiseuille flow.
Bull. Math. Biol., 36, 127-141.

MORGAN, B.E. AND YOUNG, D.F. (1974).
An integral method for the analysis of flow in arterial stenosis.
Bull. Math. Biol., 36, 39-53.

NEREM, R.M. AND LEVESQUE, M.J. (1987).
Fluid mechanics in atherosclerosis : In Handbook of Bioengineering (Eds) Richard Skalak and Shu Chien.
McGraw Hill Book Company, New York.

ODEBLAD, E. (1959).
The physics of the cervical mucus.
Acta Obstet. Gynec. Scand., 38, Suppl., 1, 44.

ODEBLAD, E. (1962).
Undulations of macro molecules in cervical mucus.
Int. J. Fert., 7, 313.

PARIHAR, R.S. (1983).
Effects of cilia and dynamical interaction on bio-fluid transport in tubules : In, Mathematics in Biology & Medicine.
An International Conference, Bari, Italy, July 18-22.

PARVATHAMMA, S. AND DEVANATHAN, R. (1983).
Microcontinuum approach for blood flow through a tube with stenosis.
Proc. of the First Int. Con. of Physio. Fluid Dynamics, Madras, India (Sept. 5-7), (Costed and Scpfd, India).
(Eds) Nigam, S.D. and Singh, M, Page 69-73.

PATEL, P.D., PICOLOGLOU, B.F. AND LYKOUDIS, P.S. (1973).
Biorheological aspects of colonic activity Part II. Experimental investigation of the rheological behaviour of human feces.
Biorheology, 10, 441-445.

PICOLOGLOU, B.F., PATEL, P.D. AND LYKOUDIS, P.S. (1973).
Biorheological aspects of colonic activity Part I. Theoretical considerations.
Biorheology, 10, 431-440.

PIRONNEAU, D. (1975).
The fluid dynamics of uninflagellated micro-organism.
Winter School in Physiological Fluid Dynamics, I.I.T. Delhi, India, 8-18 Dec.

POPEL, A.S., REGIRER, S.A. AND USICK, P.I. (1974).
A continuum model of blood flow.
Biorheology, 11, 427.

PRALHAD, R.N. AND SCHULTZ, D.H. (1988).
Two layered blood flow in stenosed tubes for different diseases.
Biorheology, 25, 715-726.

RADHAKRISHNAMACHARYA, G. (1982).
Long wavelength approximation to peristaltic motion of a power law fluid.
Rheolog. Acta. 21, 30-35.

RAJU, K.K. AND DEVANATHAN, R. (1972).
Peristaltic motion of a non-Newtonian fluid.
Rheolog. Acta., 11, 170-178.

RAJU, K.K. AND DEVANATHAN, R. (1974).
Peristaltic motion of a non-Newtonian fluid Part II. Visco-elastic fluid.
Rheol. Acta., 13, 944-948.

RATH, H.J. (1980).
Peristaltische stromungen.
Springer-Verlag, Berlin, Heidelberg, New York.

RATH, H.J. (1983).
On peristaltic flow of a non-Newtonian fluid containing rigid spherical particles.
Proc. of the First Int. Con. of Physio. Fluid Dynamics, Madras, India (Sept. 5-7), (Costed and Scpfd, India). (Eds) Nigam, S.D. and Singh, M., Page 17-23.)

REYNOLDS, A.J. (1965).
The swimming of minute organisms.
J. Fl. Mech., 23, Pt.2, 241-260.

RODKIEWICZ, C.M. (1974).
Atherosclerotic formations in the light of fluid mechanics.
Fluid Engng. Conf., Montreal, Jointly sponsored by the American and Canadian Soc. Mech. Engrs.

RODKIEWICZ, C.M. OTTO, W.J. AND SCOTT, C.W. (1979).
Empirical relationship for the flow of bile.
J. Biomech., 12, 411-413.

ROTH, A.C., YOUNG, D.F. AND CHOLVIN, N.R. (1976).
Effect of collateral and peripheral resistance on blood flow through arterial stenoses.
J. Biomech., 9, 367-375.

RUBINOW, S.I. (1975).
Introduction to mathematical biology.
Wiley, New York, 175-187.

SAGAYAMARY, R.V. AND DEVANATHAN, R. (1989).
Steady flow of couple stress fluid through tubes of slowly varying cross-sections : Application to blood flow.
Biorheology, 26, 753-769.

SANKARA SUBRAMANIAN, R. AND GILL, W.N. (1972).
Dispersion from a prescribed concentration distribution in time variable flow.
Proc. of Roy. Soc. Lond., A329, 479-492.

SCHERER, P.W., SHENDALMAN, L.H. AND GREENE, N.M. (1972).
Simultaneous diffusion and convection in single breath lung washout.
Bull. Math. Biophy., 34, 393.

SHACK, W.J. AND LARDNER, T.J. (1972).
Cilia transport.
Bull. Math. Biophys., 34, 325-335.

SHACK, W.J., FRAY, C.S. AND LARDNER, T.J. (1974).
Observations on the hydrodynamics and swimming motions of mammalian spermatozoa.
Bull. Math. Biol., 36, 555-565.

SHACK, W.J. AND LARDNER, T.J. (1974).
A long wavelength solution for a microorganism swimming in a channel.
Bull. Math. Biol., 36, 435-444.

SHAH, S.N. AND COX (1974).
Dispersion of solutes in non-Newtonian laminar flow through a circular tube : Erying model fluid.
Chem. Engg. Sci., 29, 1282.

SHAPIRO, A.H., JAFFRIN, M.Y. AND WEINBERG, S.L. (1969).
Peristaltic pumping with long wavelength at low Reynolds number.
J. Fl. Mech., 37, 799-825.

SHEN, T.S., TAM, P.Y., SHACK, W.J. AND LARDNER, T.J. (1975).
Large amplitude motion of self-propelling slender filaments at low Reynolds number.
J. Biomech., 8, 229-236.

SHEN, M.C., LIN, K.C. AND SHIH, S.M. (1981).
Peristaltic transport of a fluid-particle mixture.
SIAM, J. Math. Anal., 12, 49-59.

SHUKLA, J.B., RAO, B.R.P. AND PARIHAR, R.S. (1978).
Swimming of spermatozoa in cervix : Effects of dynamical interaction and peripheral layer viscosity.
J. Biomech. 11, 15-19.

SHUKLA, J.B., PARIHAR, R.S. AND RAO B.P.P. (1979).
Dispersion in non-Newtonian fluids : Effects of chemical reaction.
Rheol. Acta., 18, 740-748.

SHUKLA, J.B., GUPTA, S.P. AND PARIHAR, R.S. (1980a).
Biorheological aspects of blood flow through artery with mild stenosis : Effects of peripheral layer.
Biorheology, 17, 403-410.

SHUKLA, J.B., PARIHAR, R.S. AND GUPTA, S.P. (1980b).
Effects of peripheral layer viscosity on blood flow through the artery with mild stenosis.
Bull. Math. Biol. 42, 797-805.

SHUKLA, J.B., PARIHAR, R.S. AND RAO, B.R.P. (1980c).
Effects of stenosis on non-Newtonian flow of the blood in an artery.
Bull. Math. Biol., 42, 283-294.

SHUKLA, J.B. AND PARIHAR, R.S. (1980d).
Dispersion through a flowing fluid with variable viscosity and diffusion coefficient.
Phys. Fluids, 23(b), 1271.

SHUKLA, J.B., PARIHAR, R.S., RAO, B.R.P. AND GUPTA, S.P.(1980e).
Effects of peripheral layer viscosity on peristaltic transport of a bio-fluid.
J. Fl. Mech., 97, Part 2, 225-237.

SHUKLA, J.B. AND GUPTA, S.P. (1982a).
Dispersion in a Bingham plastic fluid : Effects of peripheral layer.
Rheolog. Acta, 21, 713-719.

SHUKLA, J.B. AND GUPTA, S.P. (1982b).
Peristaltic transport of a power law fluid with variable consistency.
J. Biomech. Engng., Trans. of ASME, 104, 182-186.

SHUKLA, J.B., CHANDRA, P. AND RAJIV SHARMA (1988).
Effects of peristaltic and longitudinal wave motion of the channel wall on movement of micro-organisms : Application to spermatozoa transport.
J. Biomech., 21, 947-954.

SINHA, P., SINGH, C. AND PRASAD, K.R. (1982).
A microcontinuum analysis of the self propulsion of the spermatozoa in the cervical canal.
Int. J. Engng. Sci., 20, 1037-1048.

SINHA, P. AND SING, C. (1984).
Effects of couple stresses on the blood flow through an artery with mild stenosis.
Biorheology, 21, 303-315,

SMELSER, R.E., SHACK, W.J. AND LARDNER, T.J. (1974).
The swimming of spermatozoa in an active channel.
J. Biomech., 7, 349-355.

SOLOMAN, R.L. AND HUDSON, J.L. (1967).
Heterogeneous and homogeneous reactions in a tubular reactor.
AIChE Jl., 13, 545-550.

SOUNDALGEKAR, V.M. (1970).
Dispersion of soluble matter with channel flow of micro polar fluids.
Rev. Roum. Phys., 15, 505.

SOUNDALGEKAR, V.M. AND CHATURANI, P. (1980).
Effects of couple stress on the dispersion of a soluble matter in a pipe flow of blood.
Rheolog. Acta, 19, 710-715.

SPAIN (1966).
Atherosclerosis.
Scient. Amer., 215, 49-56.

SRIVASTAVA, L.M. (1985).
Flow of couple stress fluid through stenotic blood vessels.
J. Biomech., 18, 479-485.

SRIVASTAVA, L.M. (1986).
Peristaltic transport of a couple stress fluid.
Short Communication in Rheolog. Acta., 25, 636-641.

SRIVASTAVA, L.M. AND SRIVASTAVA, V.P. (1984).
Peristaltic transport of blood : Casson Model-II.
J. Biomech., 17, 821-829.

SRIVASTAVA, L.M. AND SRIVASTAVA, V.P. (1989).
Peristaltic transport of a particle-fluid suspension.
J. of Biomech. Engng., 111, 157-165.

STEHBENS, W.E. (1982).
Hemodynamics and atherosclerosis.
Biorheology, 19, 95-101.

TAYLOR, G. (1951).
Analysis of microscopic organisms.
Proc. Roy. Soc., A209, 447-461.

TAYLOR, G.I. (1953).
Dispersion of soluble matters in solvent flowing slowly through a tube.
Proc. Roy. Soc. Lond., A219, 186-203.

TAYLOR, G.I. (1954).
Condition under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion.
Proc. Roy. Soc. Lond., A225, 473-477.

WALKER, R. (1961).
Chemical reaction and diffusion in a catalytic tubular reactor.
Phys. of Fluids, 4, 1211.

WHITMORE, R.L. (1968).
Rheology of circulation.
Oxford, Pergamon.

WILLIAMS, E.W. AND JAVADPOUR (1980).
The flow of an elastico-viscous liquid in an axisymmetric pipe of slowly varying cross-section.
J. non-Newtonian Fluid Mech., 7, 171-188.

YIN, F. AND FUNG, Y.C. (1969).
Peristaltic waves in circular cylindrical tubes.
J. Appl. Mech., 36, 579.

YOUNG, D.F. (1968).
Effect of a time-dependent stenosis on flow through a tube.
J. Engng. Ind. Trans. ASME, 90, 248-254.

YOUNG, D.F. (1979).
Fluid mechanics of arterial stenosis.
J. Biomech. Engng. Trans. ASME, 101, 157-175.

ZIEN, T.F. AND OSTRACH, S. (1970).
A long wave approximation to peristaltic motion.
J. Biomech., 3, 63-75.